# CLASSICAL FIELD THEORY. ADVANCED MATHEMATICAL FORMULATION

#### G.SARDANASHVILY

Department of Theoretical Physics, Moscow State University, 117234, Moscow, Russia

In contrast with QFT, classical field theory can be formulated in strict mathematical terms of fibre bundles, graded manifolds and jet manifolds. Second Noether theorems provide BRST extension of this classical field theory by means of ghosts and antifields for the purpose of its quantization.

Keywords: Classical field theory, gauge theory, jet manifold, Lagrangian theory, Noether theorem, Higgs field, spinor field.

#### 1 Introduction

Contemporary QFT is mainly developed as quantization of classical field models. In contrast with QFT, classical field theory can be formulated in a strict mathematical way that we present.

Observable classical fields are an electromagnetic field, Dirac spinor fields and a gravitational field on a world real smooth manifold. Their dynamic equations are Euler-Lagrange equations derived from a certain Lagrangian. One also considers classical non-abelian gauge fields and Higgs fields. Basing on these models, we develop Lagrangian theory of classical Grassmann-graded (even and odd) fields on an arbitrary smooth manifold in a very general setting. Geometry of fibre bundles is known to provide the adequate mathematical formulation of classical gauge theory and gravitation theory. Generalizing this formulation, we define even classical fields as sections of smooth fibre bundles and, accordingly, develop classical field theory as dynamic theory on fibre bundles. It is conventionally formulated in terms of jet manifolds [2, 15, 20, 38, 48, 51, 57, 78, 83].

Note that we are in the category of finite-dimensional smooth real manifolds, which are Hausdorff, second-countable and, consequently, paracompact. Let X be such a manifold. If classical fields form a projective  $C^{\infty}(X)$ -module of finite rank, their representation by sections of a fibre bundle follows from the well-known Serre–Swan theorem extended to non-compact manifolds [43].

Lagrangian theory on fibre bundles is algebraically formulated in terms of the variational bicomplex of exterior forms on jet manifolds [5, 15, 40, 42, 65, 79, 81, 83]. We are not concerned with solutions of field equations, but develop classical field theory as *sui generis* 

prequantum theory that necessarily involves odd fields. For instance, these are ghosts and antifields in the second Noether theorem.

There are different descriptions of odd fields in terms of graded manifolds [21, 61] and supermanifolds [25, 32]. Both graded manifolds and supermanifolds are described in terms of sheaves of graded commutative algebras [9, 59]. However, graded manifolds are characterized by sheaves on smooth manifolds, while supermanifolds are constructed by gluing of sheaves on supervector spaces. Treating odd fields on a smooth manifold X, we follow the Serre–Swan theorem generalized to graded manifolds [13]. This states that, if a Grassmann  $C^{\infty}(X)$ -algebra is an exterior algebra of some projective  $C^{\infty}(X)$ -module of finite rank, it is isomorphic to the algebra of graded functions on a graded manifold whose body is X.

Lagrangian theory on fibre bundles is generalized to Lagrangian theory of even and odd variables on graded manifolds in terms of the Grassmann-graded variational bicomplex [8, 12, 13, 42, 83]. Theorem 1 on cohomology of the variational bicomplex results in a solution of the global inverse problem of the calculus of variations (Theorem 2), the first variational formula (Theorem 3) and to the first Noether theorem in a very general setting of supersymmetries depending on higher-order derivatives of fields (Theorem 4).

Quantization of Lagrangian field theory essentially depends on its degeneracy, characterized by non-trivial Noether and higher-stage Noether identities, and implies its BRST extension by means of the corresponding ghosts and antifields [8, 14, 37, 47].

Any Euler-Lagrange operator satisfies Noether identities (henceforth NI) which are separated into the trivial and non-trivial ones. These NI obey first-stage NI, which in turn are subject to the second-stage NI, and so on. However, there is a problem how to select trivial and non-trivial higher-stage NI. We follow the general notion of NI of a differential operator [72]. They are represented by one-cycles of a certain chain complex. Its boundaries are trivial NI, and non-trivial NI modulo the trivial ones are given by first homology of this complex. To describe (k + 1)-stage NI, let us assume that non-trivial k-stage NI are generated by a projective  $C^{\infty}(X)$ -module  $C_{(k)}$  of finite rank and that a certain homology condition holds [13, 14, 83]. In this case, (k + 1)-stage NI are represented by (k + 2)-cycles of some chain complex of modules of antifields isomorphic to  $C_{(i)}$ ,  $i \leq k$ , by virtue of the Serre–Swan theorem. Accordingly, trivial (k + 1)-stage NI are defined as its boundaries. Iterating the arguments, we come to the exact Koszul–Tate (henceforth KT) complex (21) with the boundary KT operator (20) whose nilpotentness is equivalent to all non-trivial NI (Theorem 5) [13, 14, 83].

The inverse second Noether theorem (Theorem 6) associates to the KT complex the cochain sequence (24) with the ascent operator **u** (25), called the gauge operator. Its components (26) and (27) are non-trivial gauge and higher-stage gauge symmetries of Lagrangian theory. They obey the gauge symmetry conditions (30) and (32). The gauge operator unlike the KT one is not nilpotent, unless gauge symmetries are abelian. Therefore, in con-

trast with NI, an intrinsic definition of non-trivial gauge and higher-stage gauge symmetries meets difficulties. Defined by the gauge operator, gauge and higher-stage gauge symmetries are indexed by ghosts, not gauge parameters. Herewith, k-stage gauge symmetries act on (k-1)-stage ghosts.

Gauge symmetries fail to form an algebra in general [35, 44, 47]. We say that gauge and higher-stage gauge symmetries are algebraically closed if the gauge operator  $\mathbf{u}$  (25) admits the nilpotent BRST extension  $\mathbf{b}$  (33) where k-stage gauge symmetries are extended to k-stage BRST transformations acting both on (k-1)-stage and k-stage ghosts [44, 83]. The BRST operator (33) brings the cochain sequence (24) into the BRST complex.

The KT and BRST complexes provide an above mentioned BRST extension of original Lagrangian field theory. This extension exemplifies so called field-antifield theory whose Lagrangians are required to satisfy the particular condition (36) called the classical master equation. We show that an original Lagrangian is extended to a proper solution of the master equation if the gauge operator (25) admits a nilpotent BRST extension (Theorem 7) [14, 83].

Given the BRST operator (33), a desired proper solution of the master equation is constructed by the formula (38). This construction completes the BRST extension of original Lagrangian theory to the prequantum one, quantized in terms of functional integrals.

The basic field models, including gauge theory, gravitation theory and spinor fields, are briefly considered.

#### 2 Jet manifolds

Jet formalism [5, 38, 56, 75, 76, 79] provides the conventional language of theory of differential equations and Lagrangian theory on fibre bundles [15, 20, 38, 48, 51, 57, 78, 83].

Given a smooth fibre bundle  $Y \to X$ , a k-order jet  $j_x^k s$  at a point  $x \in X$  is defined as an equivalence class of sections s of  $Y \to X$  identified by k+1 terms of their Taylor series at x. A key point is that a set  $J^k Y$  of k-order jets is a finite-dimensional smooth manifold coordinated by  $(x^\lambda, y^i, y^i_\lambda, \dots, y^i_{\lambda_k \dots \lambda_1})$ , where  $(x^\lambda, y^i)$  are bundle coordinates on  $Y \to X$  and  $y^i_{\lambda_r \dots \lambda_1}$  are coordinates of derivatives, i.e.,  $y^i_{\lambda_r \dots \lambda_1} \circ s = \partial_{\lambda_r} \cdots \partial_{\lambda_1} s(x)$ . Accordingly, the infinite order jets are defined as equivalence classes of sections of a fibre bundle  $Y \to X$  identified by their Taylor series. Infinite order jets form a paracompact Fréchet (not smooth) manifold  $J^\infty Y$ . It coincides with the projective limit of the inverse system of finite order jet manifolds

$$X \longleftarrow Y \longleftarrow J^1 Y \longleftarrow \cdots J^{r-1} Y \longleftarrow J^r Y \longleftarrow \cdots$$
 (1)

The main advantage of jet formalism is that it enables one to deal with finite-dimensional jet manifolds instead of infinite-dimensional spaces of fields. In the framework of jet formalism, a k-order differential equation on a fibre bundle  $Y \to X$  is defined as a closed

subbundle  $\mathfrak{E}$  of the jet bundle  $J^kY \to X$ . Its solution is a section s of  $Y \to X$  whose jet prolongation  $J^ks$  lives in  $\mathfrak{E}$ . A k-order differential operator on  $Y \to X$  is defined as a morphism of the jet bundle  $J^kY \to X$  to some vector bundle  $E \to X$ . However, the kernel of a differential operator need not be a differential equation.

Jet manifolds provide the language of modern differential geometry. Due to the canonical bundle monomorphism  $J^1Y \to T^*X \otimes TY$  over Y, any connection  $\Gamma$  on a fibre bundle  $Y \to X$  is represented by a global section

$$\Gamma = dx^{\lambda} \otimes (\partial_{\alpha} + \Gamma_{\lambda}^{i}(x^{\mu}, y^{j})\partial_{i})$$

of the jet bundle  $J^1Y \to Y$ . Accordingly, we have the  $T^*X \otimes VY$ -valued first order differential operator

$$D = (y_{\alpha}^{i} - \Gamma_{\lambda}^{i}) dx^{\lambda} \otimes \partial_{i}$$

on Y. It is called the covariant differential.

Note that there are different notions of jets. Jets of sections are particular jets of maps [56] and jets of submanifolds [38, 57, 83]. Let us mention jets of modules over commutative rings [57, 59] and graded commutative rings [43], and of modules over algebras of operadic type [62]. Jets of modules over a noncommutative ring however fail to be defined.

# 3 Lagrangian theory of even fields

We formulate Lagrangian theory on fibre bundles in algebraic terms of the variational bicomplex [5, 40, 42, 75, 79, 81, 83].

The inverse system (1) of jet manifolds yields the direct system

$$\mathcal{O}^*X \longrightarrow \mathcal{O}^*Y \longrightarrow \mathcal{O}_1^*Y \longrightarrow \cdots \mathcal{O}_{r-1}^*Y \longrightarrow \mathcal{O}_r^*Y \longrightarrow \cdots$$
 (2)

of differential graded algebras (henceforth DGAs)  $\mathcal{O}_r^*Y$  of exterior forms on jet manifolds  $J^rY$ . Its direct limit is the DGA  $\mathcal{O}_\infty^*Y$  of all exterior forms on finite order jet manifolds. This DGA is locally generated by horizontal forms  $dx^\lambda$  and contact forms  $\theta_\Lambda^i = dy_\Lambda^i - y_{\lambda+\Lambda}^i dx^\lambda$ , where  $\Lambda = (\lambda_k...\lambda_1)$  denotes a symmetric multi-index, and  $\lambda + \Lambda = (\lambda \lambda_k...\lambda_1)$ . There is the canonical decomposition of  $\mathcal{O}_\infty^*Y$  into the modules  $\mathcal{O}_\infty^{k,m}Y$  of k-contact and m-horizontal forms ( $m \leq n = \dim X$ ). Accordingly, the exterior differential on  $\mathcal{O}_\infty^*Y$  falls into the sum  $d = d_V + d_H$  of the vertical differential  $d_V : \mathcal{O}_\infty^{k,*}Y \to \mathcal{O}_\infty^{k+1,*}Y$  and the total one  $d_H : \mathcal{O}_\infty^{*,m}Y \to \mathcal{O}_\infty^{*,m+1}Y$ . One also introduces the projector  $\varrho$  on  $\mathcal{O}_\infty^{>0,n}Y$  such that  $\varrho \circ d_H = 0$  and the variational operator  $\varrho \circ d$  on  $\mathcal{O}_\infty^{*,n}Y$  such that  $\varrho \circ d \circ d$  and the variational operator  $\varrho \circ d$  on  $\mathcal{O}_\infty^{*,n}Y$  such that  $\varrho \circ d \circ d$  and the variational operator  $\varrho \circ d$  on  $\mathcal{O}_\infty^{*,n}Y$  such that  $\varrho \circ d \circ d$  on  $\mathcal{O}_\infty^{*,n}Y$  into the variational bicomplex. We consider its

subcomplexes

$$0 \to \mathbb{R} \to \mathcal{O}_{\infty}^{0} Y \xrightarrow{d_{H}} \mathcal{O}_{\infty}^{0,1} Y \cdots \xrightarrow{d_{H}} \mathcal{O}_{\infty}^{0,n} Y \xrightarrow{\delta} \mathbf{E}_{1} \xrightarrow{\delta} \mathbf{E}_{2} \longrightarrow \cdots, \tag{3}$$

$$0 \to \mathcal{O}_{\infty}^{1,0} Y \xrightarrow{d_H} \mathcal{O}_{\infty}^{1,1} Y \cdots \xrightarrow{d_H} \mathcal{O}_{\infty}^{1,n} Y \xrightarrow{\varrho} \mathbf{E}_1 \to 0, \qquad \mathbf{E}_k = \varrho(\mathcal{O}_{\infty}^{k,n} Y). \tag{4}$$

Their elements  $L \in \mathcal{O}^{0,n}_{\infty}Y$  and  $\delta L \in \mathbf{E}_1$  are finite order Lagrangians on a fibre bundle  $Y \to X$  and their Euler-Lagrange operators.

The algebraic Poincaré lemma [65, 81] states that the variational bicomplex  $\mathcal{O}_{\infty}^* Y$  is locally exact. In order to obtain its cohomology, one therefore can use the abstract de Rham theorem on sheaf cohomology [52] and the fact that Y is a strong deformation retract of  $J^{\infty}Y$ , i.e., sheaf cohomology of  $J^{\infty}Y$  equals that of Y [5, 40]. A problem is that the paracompact space  $J^{\infty}Y$  admits the partition of unity by functions which do not belong to  $\mathcal{O}_{\infty}^{0}Y$ . Therefore, one considers the variational bicomplex  $\mathcal{Q}_{\infty}^{*}Y \supset \mathcal{O}_{\infty}^{*}Y$  whose elements are locally exterior forms on finite order jet manifolds, and obtains its cohomology [79, 70]. Afterwards, the  $d_{H^{-}}$  and  $\delta$ -cohomology of  $\mathcal{O}_{\infty}^{*}Y$  is proved to be isomorphic to that of  $\mathcal{Q}_{\infty}^{*}Y$  [39, 40, 70, 83]. In particular, cohomology of the variational complex (3) equals the de Rham cohomology of Y, while the complex (4) is exact.

The exactness of the complex (4) at the last term states the global first variational formula which, firstly, shows that an Euler-Lagrange operator  $\delta L$  is really a variational operator of the calculus of variations and, secondly, leads to the first Noether theorem. Cohomology of the variational complex (3) at the term  $\mathcal{O}_{\infty}^{0,n}Y$  provides a solution of the global inverse problem of the calculus of variations on fibre bundles. It is the cohomology of variationally trivial Lagrangians which are locally  $d_H$ -exact.

#### 4 Odd fields

The algebraic formulation of Lagrangian theory of even fields in terms of the variational bicomplex is generalized to odd fields [8, 12, 13, 42, 83].

Namely, let a bundle  $Y \to X$  of classical fields be a vector bundle. Then all jet bundles  $J^kY \to X$  are also vector bundles. Let us consider a subalgebra  $P^*_{\infty}Y \subset \mathcal{O}^*_{\infty}Y$  of exterior forms whose coefficients are polynomial in fibre coordinates  $y^i$ ,  $y^i_{\Lambda}$  on these bundles. In particular, the commutative ring  $P^0_{\infty}Y$  consists of polynomials of coordinates  $y^i$ ,  $y^i_{\Lambda}$  with coefficients in the ring  $C^{\infty}(X)$ . One can associate to such a polynomial a section of the symmetric tensor product  $V^0(J^kY)^*$  of the dual of the jet bundle  $J^kY \to X$ , and vice versa. Moreover, any element of  $P^*_{\infty}Y$  is an element of the Chevalley–Eilenberg differential calculus over  $P^0_{\infty}Y$ . This construction is extended to the case of odd fields.

In accordance with the Serre–Swan theorem [43], if a Grassmann  $C^{\infty}(X)$ -algebra is the exterior algebra of some projective  $C^{\infty}(X)$ -module of finite rank, it is isomorphic to the

algebra of graded functions on a graded manifold  $(X, \mathcal{A}_F)$  whose a body is X and whose structure ring  $\mathcal{A}_F$  of graded functions consists of sections of the exterior bundle

$$\wedge F^* = \mathbb{R} \oplus F^* \oplus \bigwedge^2 F^* \oplus \cdots,$$

where  $F^*$  is the dual of some vector bundle  $F \to X$ . Then the Grassmann-graded Chevalley–Eilenberg differential calculus

$$0 \to \mathbb{R} \to \mathcal{A}_F \xrightarrow{d} \mathcal{S}^1[F;X] \xrightarrow{d} \cdots \mathcal{S}^k[F;X] \xrightarrow{d} \cdots$$

over  $\mathcal{A}_F$  can be constructed. One can think of its elements as being graded differential forms on X. In particular, there is a monomorphism  $\mathcal{O}^*X \to \mathcal{S}^*[F;X]$ . Following suit of an even DGA  $P_{\infty}^*Y$ , let us consider simple graded manifolds  $(X, \mathcal{A}_{J^rF})$  modelled over the vector bundles  $J^rF \to X$ . We have the direct system of corresponding DGAs

$$S^*[F;X] \longrightarrow S^*[J^1F;X] \longrightarrow \cdots S^*[J^rF;X] \longrightarrow \cdots$$

whose direct limit  $\mathcal{S}_{\infty}^*[F;X]$  is the Grassmann counterpart of an even DGA  $P_{\infty}^*Y$ .

The total algebra of even and odd fields is the graded exterior product

$$\mathcal{P}_{\infty}^{*}[F;Y] = P_{\infty}^{*}Y \bigwedge_{\mathcal{O}^{*}X} \mathcal{S}_{\infty}^{*}[F;X]$$
(5)

of the DGAs  $P_{\infty}^*Y$  and  $\mathcal{S}_{\infty}^*[F;X]$  over their common subalgebra  $\mathcal{O}^*X$  [12, 42, 83]. In particular,  $\mathcal{P}_{\infty}^0[F;Y]$  is a graded commutative  $C^{\infty}(X)$ -ring whose even and odd generating elements are sections of  $Y \to X$  and  $F \to X$ , respectively. Let  $(x^{\lambda}, y^i, y^i_{\Lambda})$  be bundle coordinates on jet bundles  $J^kY \to X$  and  $(x^{\lambda}, c^a, c^a_{\Lambda})$  those on  $J^rF \to X$ . For simplicity, let these symbols also stand for local sections s of these bundles such that  $s^i_{\Lambda}(x) = y^i_{\Lambda}$  and  $s^a_{\Lambda}(x) = c^a_{\Lambda}$ . Then the DGA  $\mathcal{P}_{\infty}^*[F;Y]$  (5) is locally generated by elements  $(y^i, y^i_{\Lambda}, c^a, c^a_{\Lambda}, dx^{\lambda}, dy^i, dy^i_{\Lambda}, dc^a, dc^a_{\Lambda})$ . By analogy with  $(y^i, y^i_{\Lambda})$ , one can think of odd generating elements  $(c^a, c^a_{\Lambda})$  as being (local) odd fields and their jets.

In a general setting, if  $Y \to X$  is not a vector bundle, we consider graded manifolds  $(J^rY, \mathfrak{A}_{F_r})$  whose bodies are jet manifolds  $J^rY$ , and  $F_r = J^rY \times J^rF$  is the pull-back onto  $J^rY$  of the jet bundle  $J^rF \to X$  [13, 14, 83]. As a result, we obtain the direct system of DGAs

$$S^*[Y \times F; Y] \longrightarrow S^*[F_1; J^1Y] \longrightarrow \cdots S^*[F_r; J^rY] \longrightarrow \cdots.$$
(6)

Its direct limit  $\mathcal{S}^*_{\infty}[F;Y]$  is a differential calculus over the ring  $\mathcal{S}^0_{\infty}[F;Y]$  of graded functions. The monomorphisms  $\mathcal{O}^*_rY \to \mathcal{S}^*[F_r;J^rY]$  yield a monomorphism of the direct system (2) to that (6) and, consequently, the monomorphism  $\mathcal{O}^*_{\infty}Y \to \mathcal{S}^*_{\infty}[F;Y]$  of their direct limits. Moreover,  $\mathcal{S}^*_{\infty}[F;Y]$  is a  $\mathcal{O}^0_{\infty}Y$ -algebra. It contains the  $C^{\infty}(X)$ -subalgebra  $\mathcal{P}^*_{\infty}[F;Y]$  if a fibre bundle  $Y \to X$  is affine. The  $\mathcal{O}^0_{\infty}Y$ -algebra  $\mathcal{S}^*_{\infty}[F;Y]$  is locally generated by

elements  $(c^a, c^a_{\Lambda}, dx^{\lambda}, dy^i, dy^i_{\Lambda}, dc^a, dc^a_{\Lambda})$  with coefficient functions depending on coordinates  $(x^{\lambda}, y^i, y^i_{\Lambda})$ . One calls  $(y^i, c^a)$  the local basis for the DGA  $\mathcal{S}^*_{\infty}[F; Y]$ . We further use the collective symbol  $s^A$  for its elements. Accordingly,  $s^A_{\Lambda}$  denote jets of  $s^A$ ,  $\theta^A_{\Lambda} = ds^A_{\Lambda} - s^A_{\lambda+\Lambda} dx^{\lambda}$  are contact forms, and  $\partial^{\Lambda}_{A}$  are graded derivations of the  $\mathbb{R}$ -ring  $\mathcal{S}^0_{\infty}[F; Y]$  such that  $\partial^{\Lambda'}_{A'} | ds^A_{\Lambda} = \delta^A_{A'} \delta^{\Lambda'}_{\Lambda}$ . The symbol  $[A] = [s^A] = [s^A]$  stands for the Grassmann parity.

The DGA  $\mathcal{S}_{\infty}^*[F;Y] \supset \mathcal{O}_{\infty}^*Y$  is split into the variational bicomplex which describes Lagrangian theory of even and odd fields.

## 5 Lagrangian theory of even and odd fields

There is the canonical decomposition of the DGA  $\mathcal{S}_{\infty}^*[F;Y]$  into the modules  $\mathcal{S}_{\infty}^{k,m}[F;Y]$  of k-contact and m-horizontal graded forms. Accordingly, the graded exterior differential on  $\mathcal{S}_{\infty}^*[F;Y]$  falls into the sum  $d=d_V+d_H$  of the vertical differential  $d_V$  and the total differential

$$d_{H}(\phi) = dx^{\lambda} \wedge d_{\lambda}(\phi), \qquad d_{\lambda} = \partial_{\lambda} + \sum_{0 \leq |\Lambda|} s_{\lambda + \Lambda}^{A} \partial_{A}^{\Lambda}, \qquad \phi \in \mathcal{S}_{\infty}^{*}[F; Y],$$
  
$$d_{H} \circ h_{0} = h_{0} \circ d, \qquad h_{0} : \mathcal{S}_{\infty}^{*}[F; Y] \to \mathcal{S}_{\infty}^{0,*}[F; Y].$$

We also have the graded projection endomorphism  $\varrho$  of  $\mathcal{S}_{\infty}^{<0,n}[F;Y]$  such that  $\varrho \circ d_H = 0$  and the graded variational operator  $\delta = \varrho \circ d$  such that  $\delta \circ d_H = 0$ ,  $\delta \circ \delta = 0$ . With these operators the DGA  $\mathcal{S}_{\infty}^*[F;Y]$  is split into the Grassmann-graded variational bicomplex. It contains the subcomplexes

$$0 \to \mathbb{R} \longrightarrow \mathcal{S}_{\infty}^{0}[F;Y] \xrightarrow{d_{H}} \mathcal{S}_{\infty}^{0,1}[F;Y] \cdots \xrightarrow{d_{H}} \mathcal{S}_{\infty}^{0,n}[F;Y] \xrightarrow{\delta} \mathbf{E}_{1} = \varrho(\mathcal{S}_{\infty}^{1,n}[F;Y]), \tag{7}$$

$$0 \to \mathcal{S}^{1,0}_{\infty}[F;Y] \xrightarrow{d_H} \mathcal{S}^{1,1}_{\infty}[F;Y] \cdots \xrightarrow{d_H} \mathcal{S}^{1,n}_{\infty}[F;Y] \xrightarrow{\varrho} \mathbf{E}_1 \to 0.$$
 (8)

One can think of their even elements

$$L = \mathcal{L}\omega \in \mathcal{S}^{0,n}_{\infty}[F;Y], \qquad \omega = dx^1 \wedge \dots \wedge dx^n,$$
 (9)

$$\delta L = \theta^A \wedge \mathcal{E}_A \omega = \sum_{0 \le |\Lambda|} (-1)^{|\Lambda|} \theta^A \wedge d_{\Lambda}(\partial_A^{\Lambda} L) \omega \in \mathbf{E}_1$$
 (10)

as being a graded Lagrangian and its Euler-Lagrange operator, respectively.

The algebraic Poincaré lemma states that the complexes (7) and (8) are locally exact at all the terms, except  $\mathbb{R}$  [8, 42]. Then one can obtain cohomology of these complexes in the same manner as that of the complexes (3) and (4) [75, 83].

**Theorem 1.** Cohomology of the variational complex (7) equals the de Rham cohomology  $H^*(Y)$  of Y. The complex (8) is exact.

Cohomology of the complex (7) at the term  $\mathcal{S}^{1,n}_{\infty}[F;Y]$  provides the following solution of the global inverse problem of the calculus of variation for graded Lagrangians.

**Theorem 2.** A  $\delta$ -closed (i.e., variationally trivial) graded density reads  $L_0 = h_0 \psi + d_H \xi$ ,  $\xi \in \mathcal{S}^{0,n-1}_{\infty}[F;Y]$ , where  $\psi$  is a non-exact n-form on Y. In particular, a  $\delta$ -closed odd density is  $d_H$ -exact.

Exactness of the complex (8) at the last term implies that any Lagrangian L admits the decomposition

$$dL = \delta L - d_H \Xi, \qquad \Xi \in \mathcal{S}_{\infty}^{1, n-1}[F; Y], \tag{11}$$

where  $L + \Xi$  is a Lepagean equivalent of L. This decomposition leads to the first variational formula (Theorem 3) and the first Noether theorem (Theorem 4).

## 6 The first Noether theorem

In order to treat symmetries of Lagrangian field theory described by the DGA  $\mathcal{S}_{\infty}^*[F;Y]$  are defined as contact graded derivations of the  $\mathbb{R}$ -ring  $\mathcal{S}_{\infty}^0[F;Y]$  [12, 42, 83]. Its graded derivation  $\vartheta$  is called contact if the Lie derivative  $\mathbf{L}_{\vartheta}$  of the DGA  $\mathcal{S}_{\infty}^*[F;Y]$  preserves the ideal of contact graded forms. Contact graded derivations take the form

$$\vartheta = \vartheta_H + \vartheta_V = \vartheta^{\lambda} d_{\lambda} + (\upsilon^A \partial_A + \sum_{|\Lambda| > 0} d_{\Lambda} \upsilon^A \partial_A^{\Lambda}), \qquad \upsilon^A = \vartheta^A - s_{\mu}^a \vartheta^{\mu}, \tag{12}$$

where  $\vartheta^{\lambda}$ ,  $\vartheta^{A}$  are local graded functions.

**Theorem 3.** It follows from the decomposition (11) that the Lie derivative  $\mathbf{L}_{\vartheta}L$  of a Lagrangian L (9) with respect to a graded derivation  $\vartheta$  (12) fulfills the first variational formula

$$\mathbf{L}_{\vartheta}L = \vartheta_V \rfloor \delta L + d_H(h_0(\vartheta \rfloor \Xi_L)) + d_V(\vartheta_H \rfloor \omega) \mathcal{L}. \tag{13}$$

In particular, if a vertical graded derivation  $\vartheta$  is treated as an infinitesimal variation of dynamic variables, then the first variational formula (13) shows that the Euler-Lagrange equations  $\delta L = 0$  are variational equations.

A contact graded derivation  $\vartheta$  (12) is called a variational symmetry of a Lagrangian L if the Lie derivative  $\mathbf{L}_{\vartheta}L$  of L is  $d_H$ -exact. One can show that  $\vartheta$  is a variational symmetry iff its vertical part  $v_V$  (12) is well. Therefore, we further restrict our consideration to vertical contact graded derivations

$$\vartheta = (\upsilon^A \partial_A + \sum_{|\Lambda| > 0} d_\Lambda \upsilon^A \partial_A^\Lambda). \tag{14}$$

A glance at the expression (14) shows that such a derivation is an infinite jet prolongation of its first summand  $v = v^A \partial_A$ , called the generalized vector field. Substituting  $\vartheta$  (14) into the first variational formula (13), we come to the first Noether theorem.

**Theorem 4.** If  $\vartheta$  (14) is a variational symmetry of a Lagrangian L (9) (i.e.,  $\mathbf{L}_v L = d_H \sigma$ ,  $\sigma \in \mathcal{S}^{0,n-1}_{\infty}$ ), the weak conservation law

$$0 \approx d_H(h_0(\vartheta | \Xi_L) - \sigma)$$

the Noether current  $\mathfrak{J}_{\vartheta} = h_0(\vartheta \rfloor \Xi_L)$  holds on the shell  $\delta L = 0$ .

A vertical graded derivation  $\vartheta$  (14) is called nilpotent if  $\mathbf{L}_{\vartheta}(\mathbf{L}_{\vartheta}\phi) = 0$  for any horizontal graded form  $\phi \in \mathcal{S}^{0,*}_{\infty}[F;Y]$ . An even graded derivation is never nilpotent.

For the sake of simplicity, the common symbol further stands for a generalized vector field v, the contact graded derivation  $\vartheta$  (14) determined by v and the Lie derivative  $\mathbf{L}_{\vartheta}$ . We agree to call all these operators a graded derivation of the DGA  $\mathcal{S}_{\infty}^{*}[F;Y]$ .

## 7 The KT complex of Noether identities

Any Euler-Lagrange operator (10) obeys NI which are separated into trivial and non-trivial ones. Trivial NI are defined as boundaries of a certain chain complex [13]. Lagrangian theory is called degenerate if there exists non-trivial NI. They satisfy first-stage NI and so on. Thus, there is a hierarchy of reducible NI. Degenerate Lagrangian theory is said to be N-stage reducible if there exists non-trivial N-stage NI, but all (N+1)-stage NI are trivial. Under certain conditions, one can associate to degenerate Lagrangian theory the exact KT complex whose boundary operator provides all non-trivial NI (Theorem 5) [13, 14, 83]. This complex is an extension of the original DGA  $\mathcal{S}_{\infty}^*[F;Y]$  by means of antifields whose spaces are density-dual to the modules of non-trivial NI.

Let us introduce the following notation. The density dual of a vector bundle  $E \to X$  is  $\overline{E}^* = E^* \otimes \stackrel{n}{\wedge} T^*X$ . Given vector bundles  $E \to X$  and  $V \to X$ , let  $\mathcal{S}^*_{\infty}[V \times F; Y \times E]$  be the extension of the DGA  $\mathcal{S}^*_{\infty}[F; Y]$  whose additional even and odd generators are sections of  $E \to X$  and  $V \to X$ , respectively. We consider its subalgebra  $\mathcal{P}^*_{\infty}[V, F; Y, E]$  with coefficients polynomial in these new generators. Let us also assume that the vertical tangent bundle VY of Y admits the splitting  $VY = Y \times W$ , where  $W \to X$  is a vector bundle. In this case, there no fibre bundle under consideration whose transition functions vanish on the shell  $\delta L = 0$ . Let  $\overline{Y}^*$  denote the density-dual of W in this splitting.

Let L be a Lagrangian (9) and  $\delta L$  its Euler–Lagrange operator (10). In order to describe NI which  $\delta L$  satisfies, let us enlarge the DGA  $\mathcal{S}_{\infty}^*[F;Y]$  to the DGA  $\mathcal{P}_{\infty}^*[\overline{Y}^*, F; Y, \overline{F}^*]$  with the local basis  $\{s^A, \overline{s}_A\}$ ,  $[\overline{s}_A] = ([A] + 1) \mod 2$ . Its elements  $\overline{s}_A$  are called antifields of antifield number  $\operatorname{Ant}[\overline{s}_A] = 1$  [8, 47]. The DGA  $\mathcal{P}_{\infty}^*[\overline{Y}^*, F; Y, \overline{F}^*]$  is endowed with the

nilpotent right graded derivation  $\overline{\delta} = \stackrel{\leftarrow}{\partial}^A \mathcal{E}_A$ . We have the chain complex

$$0 \leftarrow \operatorname{Im} \overline{\delta} \stackrel{\overline{\delta}}{\longleftarrow} \mathcal{P}^{0,n}_{\infty}[\overline{Y}^*; F; Y; \overline{F}^*]_1 \stackrel{\overline{\delta}}{\longleftarrow} \mathcal{P}^{0,n}_{\infty}[\overline{Y}^*; F; Y; \overline{F}^*]_2$$
 (15)

of graded densities of antifield number  $\leq 2$ . Its one-cycles define the above mentioned NI, which are trivial iff cycles are boundaries. Accordingly, elements of the first homology  $H_1(\overline{\delta})$  of the complex (15) correspond to non-trivial NI modulo the trivial ones [13, 14, 72, 83]. We assume that  $H_1(\overline{\delta})$  is finitely generated. Namely, there exists a projective Grassmann-graded  $C^{\infty}(X)$ -module  $C_{(0)} \subset H_1(\overline{\delta})$  of finite rank with a local basis  $\{\Delta_r\}$  such that all non-trivial NI result from the NI

$$\overline{\delta}\Delta_r = \sum_{0 < |\Lambda|} \Delta_r^{A,\Lambda} d_{\Lambda} \mathcal{E}_A = 0. \tag{16}$$

The NI (16) need not be independent, but obey first-stage NI described as follows. By virtue of the Serre–Swan theorem, the module  $C_{(0)}$  is isomorphic to a module of sections of the product  $\overline{V}^* \times \overline{E}^*$ , where  $\overline{V}^*$  and  $\overline{E}^*$  are the density-duals of some vector bundles  $V \to X$  and  $E \to X$ . Let us enlarge the DGA  $\mathcal{P}_{\infty}^*[\overline{Y}^*, F; Y, \overline{F}^*]$  to the DGA  $\mathcal{P}_{\infty}^*[\overline{E}^* \times \overline{Y}^*, F; Y, \overline{F}^* \times \overline{V}^*]$  possessing the local basis  $\{s^A, \overline{s}_A, \overline{c}_r\}$  of Grassmann parity  $[\overline{c}_r] = ([\Delta_r] + 1) \mod 2$  and antifield number  $\operatorname{Ant}[\overline{c}_r] = 2$ . This DGA is provided with the nilpotent right graded derivation  $\delta_0 = \overline{\delta} + \overleftarrow{\partial}^{-r} \Delta_r$  such that its nilpotency condition is equivalent to the NI (16). Then we have the chain complex

$$0 \leftarrow \operatorname{Im} \overline{\delta} \xleftarrow{\overline{\delta}} \mathcal{P}_{\infty}^{0,n} [\overline{Y}^*, F; Y, \overline{F}^*]_1 \xleftarrow{\delta_0} \mathcal{P}_{\infty}^{0,n} [\overline{E}^* \times \overline{Y}^*, F; Y, \overline{F}^* \times \overline{V}^*]_2$$

$$\xleftarrow{\delta_0} \mathcal{P}_{\infty}^{0,n} [\overline{E}^* \times \overline{Y}^*, F; Y, \overline{F}^* \times \overline{V}^*]_3$$

$$(17)$$

of graded densities of antifield number  $\leq 3$ . It has the trivial homology  $H_0(\overline{\delta}_0)$  and  $H_1(\overline{\delta}_0)$ . The two-cycles of this complex define the above mentioned first-stage NI. They are trivial if cycles are boundaries, but the converse need not be true, unless a certain homology condition holds [13, 14, 72, 83]. If the complex (17) obeys this condition, elements of its second homology  $H_2(\overline{\delta}_0)$  define non-trivial first-stage NI modulo the trivial ones. Let us assume that  $H_2(\delta_0)$  is finitely generated. Namely, there exists a projective Grassmann-graded  $C^{\infty}(X)$ -module  $C_{(1)} \subset H_2(\delta_0)$  of finite rank with a local basis  $\{\Delta_{r_1}\}$  such that all non-trivial first-stage NI follow from the equalities

$$\sum_{0 \le |\Lambda|} \Delta_{r_1}^{r,\Lambda} d_{\Lambda} \Delta_r + \overline{\delta} h_{r_1} = 0. \tag{18}$$

The first-stage NI (18) need not be independent, but satisfy the second-stage ones, and so on. Iterating the arguments, we come to the following [13, 14, 83].

**Theorem 5.** One can associate to degenerate N-stage reducible Lagrangian theory the exact KT complex (21) with the boundary operator (20) whose nilpotency property restarts all NI and higher-stage NI (16) and (22) if these identities are finitely generated and iff this complex obeys the homology regularity condition.

Namely, there are vector bundles  $V_1, \ldots, V_N, E_1, \ldots, E_N$  over X and the DGA

$$\overline{\mathcal{P}}_{\infty}^*\{N\} = \mathcal{P}_{\infty}^*[\overline{E}_N^* \times \dots \times \overline{E}_1^* \times \overline{E}^* \times \overline{Y}^*, F; Y, \overline{F}^* \times \overline{V}^* \times \overline{V}_1^* \times \dots \times \overline{V}_N^*]$$
(19)

with a local basis  $\{s^A, \overline{s}_A, \overline{c}_r, \overline{c}_{r_1}, \dots, \overline{c}_{r_N}\}$  of antifield number  $\operatorname{Ant}[\overline{c}_{r_k}] = k + 2$ . Let the indexes k = -1, 0 further stand for  $\overline{s}_A$  and  $\overline{c}_r$ , respectively. The DGA  $\overline{\mathcal{P}}_{\infty}^*\{N\}$  (19) is provided with the nilpotent right graded derivation

$$\delta_{N} = \overleftarrow{\partial}^{A} \mathcal{E}_{A} + \sum_{0 \leq |\Lambda|} \overleftarrow{\partial}^{r} \Delta_{r}^{A,\Lambda} \overline{s}_{\Lambda A} + \sum_{1 \leq k \leq N} \overleftarrow{\partial}^{r_{k}} \Delta_{r_{k}}, \tag{20}$$

$$\Delta_{r_{k}} = \sum_{0 \leq |\Lambda|} \Delta_{r_{k}}^{r_{k-1},\Lambda} \overline{c}_{\Lambda r_{k-1}} + \sum_{0 \leq |\Sigma|,|\Xi|} (h_{r_{k}}^{(r_{k-2},\Sigma)(A,\Xi)} \overline{c}_{\Sigma r_{k-2}} \overline{s}_{\Xi A} + \ldots),$$

of antifield number -1. It is called the KT differential. With  $\delta_N$ , we have the exact chain complex

$$0 \leftarrow \operatorname{Im} \overline{\delta} \stackrel{\overline{\delta}}{\longleftarrow} \mathcal{P}_{\infty}^{0,n} [\overline{Y}^*, F; Y, \overline{F}^*]_1 \stackrel{\delta_0}{\longleftarrow} \overline{\mathcal{P}}_{\infty}^{0,n} \{0\}_2 \stackrel{\delta_1}{\longleftarrow} \overline{\mathcal{P}}_{\infty}^{0,n} \{1\}_3 \cdots$$

$$\stackrel{\delta_{N-1}}{\longleftarrow} \overline{\mathcal{P}}_{\infty}^{0,n} \{N-1\}_{N+1} \stackrel{\delta_N}{\longleftarrow} \overline{\mathcal{P}}_{\infty}^{0,n} \{N\}_{N+2} \stackrel{\delta_N}{\longleftarrow} \overline{\mathcal{P}}_{\infty}^{0,n} \{N\}_{N+3},$$

$$(21)$$

of graded densities of antifield number  $\leq N+3$  which is assumed to satisfy the homology regularity condition. This condition states that any  $\delta_{k< N-1}$ -cycle  $\phi \in \overline{\mathcal{P}}_{\infty}^{0,n}\{k\}_{k+3} \subset \overline{\mathcal{P}}_{\infty}^{0,n}\{k+1\}_{k+3}$  is a  $\delta_{k+1}$ -boundary. The nilpotency property of the boundary operator  $\delta_N$  (20) implies the NI (16) and the  $(k \leq N)$ -stage NI

$$\sum_{0 \le |\Lambda|} \Delta_{r_k}^{r_{k-1},\Lambda} d_{\Lambda} \left( \sum_{0 \le |\Sigma|} \Delta_{r_{k-1}}^{r_{k-2},\Sigma} \overline{c}_{\Sigma r_{k-2}} \right) + \overline{\delta} \left( \sum_{0 \le |\Sigma|,|\Xi|} h_{r_k}^{(r_{k-2},\Sigma)(A,\Xi)} \overline{c}_{\Sigma r_{k-2}} \overline{s}_{\Xi A} \right) = 0. \tag{22}$$

#### 8 The inverse second Noether theorem

Second Noether theorems in different variants relate the NI and higher-stage NI to the gauge and higher-stage gauge symmetries of Lagrangian theory [11, 12, 36]. However, the notion of gauge symmetry of Lagrangian theory meets difficulties. In particular, it may happen that gauge symmetries are not assembled into an algebra, or they form an algebra on-shell [35, 44, 47, 83]. At the same time, NI are well defined (Theorem 5). Therefore, one can use the inverse second Noether theorem (Theorem 6) in order to obtain gauge symmetries of degenerate Lagrangian theory. This theorem associates to the antifield KT

complex (21) the cochain sequence (24) of ghosts, whose ascent operator (25) provides gauge and higher-stage gauge symmetries of a Lagrangian field theory.

Given the DGA  $\overline{\mathcal{P}}_{\infty}^*\{N\}$  (19), let us consider the DGA

$$\mathcal{P}_{\infty}^*\{N\} = \mathcal{P}_{\infty}^*[V_N \times \cdots V_1 \times V, F; Y, E \times E_1 \times \cdots \times E_N]$$
 (23)

possessing the local basis  $\{s^A, c^r, c^{r_1}, \dots, c^{r_N}\}$  of Grassmann parity  $[c^{r_k}] = ([\overline{c}_{r_k}] + 1) \mod 2$  and antifield number  $\operatorname{Ant}[c^{r_k}] = -(k+1)$ . Its elements  $c^{r_k}$ ,  $k \in \mathbb{N}$ , are called the ghosts of ghost number  $\operatorname{gh}[c^{r_k}] = k+1$  [8, 47].

**Theorem 6.** Given the KT complex (21), the graded commutative ring  $\mathcal{P}^0_{\infty}\{N\}$  is split into the cochain sequence

$$0 \to \mathcal{S}_{\infty}^{0}[F;Y] \xrightarrow{\mathbf{u}} \mathcal{P}_{\infty}^{0}\{N\}_{1} \xrightarrow{\mathbf{u}} \mathcal{P}_{\infty}^{0}\{N\}_{2} \xrightarrow{\mathbf{u}} \cdots, \tag{24}$$

with the odd ascent operator

$$\mathbf{u} = u + \sum_{1 \le k \le N} u_{(k)},\tag{25}$$

$$u = u^A \frac{\partial}{\partial s^A}, \qquad u^A = \sum_{0 \le |\Lambda|} c_\Lambda^r \eta(\Delta_r^A)^\Lambda,$$
 (26)

$$u_{(k)} = u^{r_{k-1}} \frac{\partial}{\partial c^{r_{k-1}}}, \qquad u^{r_{k-1}} = \sum_{0 \le |\Lambda|} c_{\Lambda}^{r_k} \eta(\Delta_{r_k}^{r_{k-1}})^{\Lambda}, \qquad k = 1, \dots, N,$$
 (27)

$$\eta(f)^{\Lambda} = \sum_{0 < |\Sigma| < k - |\Lambda|} (-1)^{|\Sigma + \Lambda|} C_{|\Sigma + \Lambda|}^{|\Sigma|} d_{\Sigma} f^{\Sigma + \Lambda}, \qquad C_b^a = \frac{b!}{a!(b-a)!}.$$

The components u (26),  $u_{(k)}$  (27) of the gauge operator  $\mathbf{u}$  (25) are the above mentioned gauge and higher-stage gauge symmetries of reducible Lagrangian theory, respectively. Indeed, let us consider the total DGA  $P_{\infty}^*\{N\}$  generated by original fields, ghosts and antifields

$$\{s^A, c^r, c^{r_1}, \dots, c^{r_N}, \overline{s}_A, \overline{c}_r, \overline{c}_{r_1}, \dots, \overline{c}_{r_N}\}.$$
 (28)

It contains subalgebras  $\overline{\mathcal{P}}_{\infty}^*\{N\}$  (19) and  $\mathcal{P}_{\infty}^*\{N\}$  (23), whose derivations  $\delta_N$  (20) and **u** (25) are prolonged to  $P_{\infty}^*\{N\}$ . Let us extend an original Lagrangian L to the Lagrangian

$$L_e = \mathcal{L}_e \omega = L + L_1 = L + \sum_{0 \le k \le N} c^{r_k} \Delta_{r_k} \omega = L + \delta_N \left( \sum_{0 \le k \le N} c^{r_k} \overline{c}_{r_k} \omega \right)$$
 (29)

of zero antifield number. It is readily observed that the KT differential  $\delta_N$  is a variational symmetry of the Lagrangian  $L_e$  (29), i.e., we have the equalities

$$\frac{\overleftarrow{\delta}(c^r \Delta_r)}{\delta \overline{s}_A} \mathcal{E}_A \omega = u^A \mathcal{E}_A \omega = d_H \sigma_0, \tag{30}$$

$$\left[\frac{\overleftarrow{\delta}(c^{r_i}\Delta_{r_i})}{\delta\overline{s}_A}\mathcal{E}_A + \sum_{k < i} \frac{\overleftarrow{\delta}(c^{r_i}\Delta_{r_i})}{\delta\overline{c}_{r_k}}\Delta_{r_k}\right]\omega = d_H\sigma_i, \qquad i = 1, \dots, N.$$
(31)

A glance at the equality (30) shows that the graded derivation u (26) is a variational symmetry of an original Lagrangian L. Parameterized by ghosts  $c^r$ , it is a gauge symmetry of L [12, 42]. The equalities (31) are brought into the form

$$\sum_{0 \le |\Sigma|} d_{\Sigma} u^{r_{i-1}} \frac{\partial}{\partial c_{\Sigma}^{r_{i-1}}} u^{r_{i-2}} = \overline{\delta}(\alpha^{r_{i-2}}), \qquad \alpha^{r_{i-2}} = -\sum_{0 \le |\Sigma|} \eta(h_{r_i}^{(r_{i-2})(A,\Xi)})^{\Sigma} d_{\Sigma}(c^{r_i} \overline{s}_{\Xi A}). \tag{32}$$

It follows that graded derivations  $u_{(k)}$  (27) are the k-stage gauge symmetries of reducible Lagrangian theory [11, 12, 44].

We agree to call  $\mathbf{u}$  (25) the gauge operator. In contrast with the KT one, this operator need not be nilpotent. We say that gauge and higher-stage gauge symmetries of Lagrangian theory are algebraically closed if the gauge operator  $\mathbf{u}$  can be extended to nilpotent graded derivation

$$\mathbf{b} = \mathbf{u} + \xi = u^{A} \partial_{A} + \sum_{1 \le k \le N} (u^{r_{k-1}} + \xi^{r_{k-1}}) \partial_{r_{k-1}} + \xi^{r_{N}} \partial_{r_{N}}$$
(33)

of ghost number 1 where the coefficients  $\xi^{r_{k-1}}$  are at least quadratic in ghosts [10, 44]. This extension is called the BRST operator. It brings the cochain sequence (24) into the BRST complex.

## 9 BRST extended field theory

Lagrangian field theory extended to ghosts and antifields exemplifies so called field-antifield Lagrangian theories of the following type [14, 83].

Given a fibre bundle  $Z \to X$  and a vector bundle  $Z' \to X$ , let us consider a DGA  $\mathcal{P}_{\infty}^*[\overline{Z}^*, Z'; Z, \overline{Z}'^*]$  with a local basis  $\{z^a, \overline{z}_a\}$ , where  $[\overline{z}_a] = ([z^a] + 1) \text{mod } 2$ . One can think of its elements  $z^a$  and  $\overline{z}_a$  as being fields and antifields, respectively. Its submodule  $\mathcal{P}_{\infty}^{0,n}[\overline{Z}^*, Z'; Z, \overline{Z}'^*]$  of horizontal densities is provided with the binary operation

$$\{\mathfrak{L}\omega,\mathfrak{L}'\omega\} = \left[\frac{\overleftarrow{\delta}\,\mathfrak{L}}{\delta\overline{z}_a}\frac{\delta\mathfrak{L}'}{\delta z^a} + (-1)^{[\mathfrak{L}][\mathfrak{L}']}\frac{\overleftarrow{\delta}\,\mathfrak{L}'}{\delta\overline{z}_a}\frac{\delta\mathfrak{L}}{\delta z^a}\right]\omega,\tag{34}$$

called the antibracket by analogy with that in field-antifield BRST theory [47]. One treats this operation as *sui generis* odd Poisson structure [1, 7]. Let us associate to a Lagrangian  $\mathfrak{L}\omega$  the odd graded derivations

$$v_{\mathfrak{L}} = \frac{\overleftarrow{\delta} \, \mathfrak{L}}{\delta \overline{z}_a} \frac{\partial}{\partial z^a}, \qquad \overline{v}_{\mathfrak{L}} = \frac{\overleftarrow{\partial}}{\partial \overline{z}_a} \frac{\delta \mathfrak{L}}{\delta z^a}. \tag{35}$$

Then the following conditions are equivalent: (i) the graded derivation  $v_{\mathfrak{L}}$  (35) is a variational symmetry of a Lagrangian  $\mathfrak{L}\omega$ , (ii) so is the graded derivation  $\overline{v}_{\mathfrak{L}}$ , (iii) the relation

$$\{\mathfrak{L}\omega,\mathfrak{L}\omega\} = 2\frac{\overleftarrow{\delta}\,\mathfrak{L}}{\delta\overline{z}_a}\frac{\delta\mathfrak{L}}{\delta z^a}\omega = d_H\sigma\tag{36}$$

holds [14]. This relation is called the (classical) master equation.

Let us consider an original Lagrangian L and its extension  $L_e$  (29), together with the odd graded derivations (35) which read

$$\upsilon_e = \frac{\overleftarrow{\delta} \mathcal{L}_1}{\delta \overline{s}_A} \frac{\partial}{\partial s^A} + \sum_{0 < k < N} \frac{\overleftarrow{\delta} \mathcal{L}_1}{\delta \overline{c}_{r_k}} \frac{\partial}{\partial c^{r_k}}, \qquad \overline{\upsilon}_e = \frac{\overleftarrow{\partial}}{\partial \overline{s}_A} \frac{\delta \mathcal{L}_1}{\delta s^A} + [\frac{\overleftarrow{\partial}}{\partial \overline{s}_A} \frac{\delta \mathcal{L}}{\delta s^A} + \sum_{0 < k < N} \frac{\overleftarrow{\partial}}{\partial \overline{c}_{r_k}} \frac{\delta \mathcal{L}_1}{\delta c^{r_k}}].$$

An original Lagrangian L trivially satisfies the master equation. A goal is to extend it to a nontrivial solution

$$L + L_1 + L_2 + \dots = L_e + L' \tag{37}$$

of the master equation by means of terms  $L_i$  of polynomial degree i > 1 in ghosts and of zero antifield number. Such an extension need not exists. One can show the following [14, 83].

**Theorem 7.** If the gauge operator **u** (25) admits the BRST extension **b** (33), the Lagrangian

$$L_E = L_e + \sum_{1 \le k \le N} \xi^{r_{k-1}} \overline{c}_{r_{k-1}} \omega = L + \mathbf{b} \left( \sum_{0 \le k \le N} c^{r_{k-1}} \overline{c}_{r_{k-1}} \right) \omega + d_H \sigma$$
 (38)

satisfies the master equation  $\{L_E, L_E\} = 0$ .

The proof of Theorem 7 gives something more. The gauge operator (25) admits the BRST extension (33) only if the higher-stage gauge symmetry conditions hold off-shell. For instance, this is the case of irreducible and abelian reducible Lagrangian theories. In abelian reducible theories, the gauge operator  $\mathbf{u}$  itself is nilpotent. In irreducible Lagrangian theory, the gauge operator admits a nilpotent BRST extension if gauge transformations form a Lie algebra.

# 10 Gauge theory of principal connections

Let us consider gauge theory of principal connections on a principal bundle  $P \to X$  with a structure Lie group G. Principal connections are G-equivariant connections on  $P \to X$  and, therefore, they are represented by sections of the quotient bundle

$$C = J^1 P/G \to X \tag{39}$$

[38, 59, 83]. This is an affine bundle coordinated by  $(x^{\lambda}, a_{\lambda}^{r})$  such that, given a section A of  $C \to X$ , its components  $A_{\lambda}^{r} = a_{\lambda}^{r} \circ A$  are coefficients of the familiar local connection form (i.e., gauge potentials). Therefore, one calls C (39) the bundle of principal connections. A key point is that its first order jet manifold  $J^{1}C$  admits the canonical splitting over C given by the coordinate expression

$$a_{\lambda\mu}^{r} = \frac{1}{2}\mathcal{F}_{\lambda\mu}^{r} + \frac{1}{2}\mathcal{S}_{\lambda\mu}^{r} = \frac{1}{2}(a_{\lambda\mu}^{r} + a_{\mu\lambda}^{r} - c_{pq}^{r}a_{\lambda}^{p}a_{\mu}^{q}) + \frac{1}{2}(a_{\lambda\mu}^{r} - a_{\mu\lambda}^{r} + c_{pq}^{r}a_{\lambda}^{p}a_{\mu}^{q}), \tag{40}$$

where  $c_{pq}^r$  are the structure constants of the Lie algebra  $\mathfrak{g}$  of G, and  $F_{\lambda\mu}^r = \mathcal{F}_{\lambda\mu}^r \circ J^1 A$  is the strength of a principal connection A.

There is a unique (Yang–Mills) quadratic gauge invariant Lagrangian  $L_{YM}$  on  $J^1C$  which factorizes through the component  $\mathcal{F}^r_{\lambda\mu}$  of the splitting (40). Its gauge symmetries are G-invariant vertical vector fields on P. They are given by sections  $\xi = \xi^r e_r$  of the Lie algebra bundle  $V_G P = VP/G$ , and define vector fields

$$\xi = (-c_{ii}^r \chi^j a_\lambda^i + \partial_\lambda \chi^r) \partial_r^\lambda \tag{41}$$

on the bundle of principal connections C such that  $\mathbf{L}_{J^1\chi}L_{YM}=0$ . The corresponding irreducible NI read

$$c_{ji}^r a_{\lambda}^i \mathcal{E}_r^{\lambda} + d_{\lambda} \mathcal{E}_j^{\lambda} = 0.$$

As a consequence, the basis  $(a_{\lambda}^r, c^r, \overline{a}_r^{\lambda}, \overline{c}_r)$  for the BRST extended gauge theory consists of gauge potentials  $a_{\lambda}^r$ , ghosts  $c^r$  of ghost number 1, and antifields  $\overline{a}_r^{\lambda}$ ,  $\overline{c}_r$  of antifield numbers 1 and 2, respectively. Replacing gauge parameters  $\chi^r$  in  $\chi$  (41) with odd ghost  $c^r$ , we obtain the gauge operator  $\mathbf{u}$  (25), whose nilpotent extension is the well known BRST operator

$$\mathbf{b} = (-c_{ji}^r c^j a_\lambda^i + c_\lambda^r) \frac{\partial}{\partial a_\lambda^r} - \frac{1}{2} c_{ij}^r c^i c^j \frac{\partial}{\partial c^r}.$$

Hence, the Yang-Mills Lagrangian is extended to a solution of the master equation

$$L_E = L_{YM} + (-c_{ij}^r c^j a_{\lambda}^i + c_{\lambda}^r) \overline{a}_r^{\lambda} \omega - \frac{1}{2} c_{ij}^r c^i c^j \overline{c}_r \omega.$$

# 11 Topological Chern–Simons gauge theory

Gauge symmetries of topological Chern–Simons (henceforth CS) gauge theory are wider than those of the Yang–Mills gauge one.

One usually considers CS theory whose Lagrangian is the local CS form derived from the local transgression formula for the second Chern characteristic form. The global CS Lagrangian is well defined, but depends on a background gauge potential [18, 19, 31, 41].

The fibre bundle  $J^1P \to C$  is a trivial G-principal bundle canonically isomorphic to  $C \times P \to C$ . This bundle admits the canonical principal connection

$$\mathcal{A} = dx^{\lambda} \otimes (\partial_{\lambda} + a_{\lambda}^{p} \varepsilon_{p}) + da_{\lambda}^{r} \otimes \partial_{r}^{\lambda}$$

[59]. Its curvature defines the canonical  $V_GP$ -valued 2-form

$$\mathfrak{F} = \left(da^r_\mu \wedge dx^\mu + \frac{1}{2}c^r_{pq}a^p_\lambda a^q_\mu dx^\lambda \wedge dx^\mu\right) \otimes e_r \tag{42}$$

on C. Given a section A of  $C \to X$ , the pull-back

$$F_A = A^* \mathfrak{F} = \frac{1}{2} F_{\lambda\mu}^r dx^\lambda \wedge dx^\mu \otimes e_r \tag{43}$$

of  $\mathfrak{F}$  onto X is the strength form of a gauge potential A.

Let  $I_k(e) = b_{r_1...r_k}e^{r_1}\cdots e^{r_k}$  be a *G*-invariant polynomial of degree k > 1 on the Lie algebra  $\mathfrak{g}$ . With  $\mathfrak{F}$  (42), one can associate to  $I_k$  the closed gauge-invariant 2k-form

$$P_{2k}(\mathfrak{F}) = b_{r_1 \dots r_k} \mathfrak{F}^{r_1} \wedge \dots \wedge \mathfrak{F}^{r_k}$$

on C. Given a section B of  $C \to X$ , the pull-back  $P_{2k}(F_B) = B^*P_{2k}(\mathfrak{F})$  of  $P_{2k}(\mathfrak{F})$  is a closed characteristic form on X. Let the same symbol stand for its pull-back onto C. Since  $C \to X$  is an affine bundle and the de Rham cohomology of C equals that of X, the forms  $P_{2k}(\mathfrak{F})$  and  $P_{2k}(F_B)$  possess the same cohomology class  $[P_{2k}(\mathfrak{F})] = [P_{2k}(F_B)]$  for any principal connection B. Thus,  $I_k(e) \mapsto [P_{2k}(F_B)] \in H^*(X)$  is the familiar Weil homomorphism. Furthermore, we obtain the transgression formula

$$P_{2k}(\mathfrak{F}) - P_{2k}(F_B) = d\mathfrak{S}_{2k-1}(a, B) \tag{44}$$

on C [41, 83]. Its pull-back by means of a section A of  $C \to X$  gives the transgression formula

$$P_{2k}(F_A) - P_{2k}(F_B) = d\mathfrak{S}_{2k-1}(A, B)$$

on X. For instance, if  $P_{2k}(\mathfrak{F})$  is the characteristic Chern 2k-form, then  $\mathfrak{S}_{2k-1}(a,B)$  is the CS (2k-1)-form.

In particular, one can choose the local section B = 0. Then,  $\mathfrak{S}_{2k-1}(a,0)$  is the local CS form. Let  $\mathfrak{S}_{2k-1}(A,0)$  be its pull-back onto X by means of a section A of  $C \to X$ . Then the CS form  $\mathfrak{S}_{2k-1}(a,B)$  (44) admits the decomposition

$$\mathfrak{S}_{2k-1}(a,B) = \mathfrak{S}_{2k-1}(a,0) - \mathfrak{S}_{2k-1}(B,0) + dK_{2k-1}. \tag{45}$$

The transgression formula (44) also yields the transgression formula

$$P_{2k}(\mathcal{F}) - P_{2k}(F_B) = d_H(h_0 \mathfrak{S}_{2k-1}(a, B)),$$

$$h_0 \mathfrak{S}_{2k-1}(a, B) = k \int_0^1 \mathcal{P}_{2k}(t, B) dt,$$

$$\mathcal{P}_{2k}(t, B) = b_{r_1 \dots r_k} (a_{\mu_1}^{r_1} - B_{\mu_1}^{r_1}) dx^{\mu_1} \wedge \mathcal{F}^{r_2}(t, B) \wedge \dots \wedge \mathcal{F}^{r_k}(t, B),$$

$$\mathcal{F}^{r_j}(t, B) = \frac{1}{2} [t a_{\lambda_j \mu_j}^{r_j} + (1 - t) \partial_{\lambda_j} B_{\mu_j}^{r_j} - t a_{\mu_j \lambda_j}^{r_j} - (1 - t) \partial_{\mu_j} B_{\lambda_j}^{r_j} + \frac{1}{2} c_{pq}^{r_j} (t a_{\lambda_j}^p + (1 - t) B_{\lambda_j}^p) (t a_{\mu_j}^q + (1 - t) B_{\mu_j}^q] dx^{\lambda_j} \wedge dx^{\mu_j} \otimes e_r,$$

$$(46)$$

on  $J^1C$ . If  $2k-1 = \dim X$ , the density  $L_{CS}(B) = h_0\mathfrak{S}_{2k-1}(a,B)$  (46) is the global CS Lagrangian of topological CS theory. The decomposition (45) induces the decomposition

$$L_{CS}(B) = h_0 \mathfrak{S}_{2k-1}(a,0) - \mathfrak{S}_{2k-1}(B,0) + d_H h_0 K_{2k-1}. \tag{47}$$

For instance, if dim X = 3, the global CS Lagrangians reads

$$L_{CS}(B) = \left[\frac{1}{2}h_{mn}\varepsilon^{\alpha\beta\gamma}a_{\alpha}^{m}(\mathcal{F}_{\beta\gamma}^{n} - \frac{1}{3}c_{pq}^{n}a_{\beta}^{p}a_{\gamma}^{q})\right]\omega - \left[\frac{1}{2}h_{mn}\varepsilon^{\alpha\beta\gamma}B_{\alpha}^{m}(F(B)_{\beta\gamma}^{n} - \frac{1}{3}c_{pq}^{n}B_{\beta}^{p}B_{\gamma}^{q})\right]\omega - d_{\alpha}(h_{mn}\varepsilon^{\alpha\beta\gamma}a_{\beta}^{m}B_{\gamma}^{n})\omega,$$

where  $\varepsilon^{\alpha\beta\gamma}$  is the skew-symmetric Levi-Civita tensor.

Since the density  $-\mathfrak{S}_{2k-1}(B,0) + d_H h_0 K_{2k-1}$  is variationally trivial, the global CS Lagrangian (47) possesses the same NI and gauge symmetries as the local one  $L_{CS} = h_0 \mathfrak{S}_{2k-1}(a,0)$ . They are the following.

In contrast with the Yang–Mills Lagrangian, the CS one  $L_{CS}(B)$  is independent of a metric on X. Therefore, its gauge symmetries are all G-invariant vector fields on a principal bundle P. They are identified to sections

$$v_P = \tau^{\lambda} \partial_{\lambda} + \chi^r e_r$$

of the vector bundle  $T_GP = TP/G \rightarrow X$ , and yield vector fields

$$v_C = \tau^{\lambda} \partial_{\lambda} + \left( -c_{pq}^r \chi^p a_{\lambda}^q + \partial_{\lambda} \chi^r - a_{\mu}^r \partial_{\lambda} \tau^{\mu} \right) \partial_r^{\lambda}$$
(48)

on the bundle of principal connections C [59, 83]. One can show that they are variational and, consequently, gauge symmetries of the global CS Lagrangian  $L_{CS}(B)$ . The vertical part

$$v_V = \left(-c_{pq}^r \chi^p a_\lambda^q + \partial_\lambda \chi^r - a_\mu^r \partial_\lambda \tau^\mu - \tau^\mu a_{\mu\lambda}^r\right) \partial_r^\lambda \tag{49}$$

of vector fields  $v_C$  (48) is also a variational symmetry of  $L_{CS}(B)$ .

As a consequence, the basis  $(a_{\lambda}^r, c^{\lambda}, c^r, \overline{a}_{r}^{\lambda}, \overline{c}_{\lambda}, \overline{c}_{r})$  of BRST extended CS theory consists of even fields  $a_{\lambda}^r$ , ghosts  $c^{\lambda}$ ,  $c^r$  and antifields  $\overline{a}_{r}^{\lambda}$ ,  $\overline{c}_{\lambda}$ ,  $\overline{c}_{r}$ . Substituting the ghosts  $c^{\lambda}$ ,  $c^r$  for gauge parameters in the vector field  $v_V$  (49), we obtain the gauge operator

$$\mathbf{u} = \left(-c_{pq}^r c^p a_\lambda^q + c_\lambda^r - c_\lambda^\mu a_\mu^r - c^\mu a_{\mu\lambda}^r\right) \partial_r^\lambda. \tag{50}$$

The corresponding irreducible NI read

$$-c_{ii}^r a_{\lambda}^i \mathcal{E}_r^{\lambda} - d_{\lambda} \mathcal{E}_i^{\lambda} = 0, \qquad -a_{\mu\lambda}^r \mathcal{E}_r^{\lambda} + d_{\lambda} (a_{\mu}^r \mathcal{E}_r^{\lambda}) = 0.$$

The gauge operator (50) admits the nilpotent BRST extension

$$\mathbf{b} = (-c_{ji}^r c^j a_\lambda^i + c_\lambda^r - c_\lambda^\mu a_\mu^r - c^\mu a_{\mu\lambda}^r) \frac{\partial}{\partial a_\lambda^r} - \frac{1}{2} c_{ij}^r c^i c^j \frac{\partial}{\partial c^r} + c_\mu^\lambda c^\mu \frac{\partial}{\partial c^\lambda}.$$

Accordingly, the CS Lagrangian is extended to the proper solution of the master equation

$$L_E = L_{CS} + \left[ \left( -c_{ij}^r c^j a_\lambda^i + c_\lambda^r \right) \overline{a}_r^\lambda - \frac{1}{2} c_{ij}^r c^i c^j \overline{c}_r + c_\mu^\lambda c^\mu \overline{c}_\lambda \right] \omega.$$

## 12 Field theory on composite bundles

Let us consider a composite fibre bundle

$$Y \to \Sigma \to X,$$
 (51)

where  $\pi_{Y\Sigma}: Y \to \Sigma$  and  $\pi_{\Sigma X}: \Sigma \to X$  are fibre bundles. It is provided with fibred coordinates  $(x^{\lambda}, \sigma^m, y^i)$ , where  $(x^{\mu}, \sigma^m)$  are bundle coordinates on  $\Sigma \to X$ , i.e., the transition functions of coordinates  $\sigma^m$  are independent of coordinates  $y^i$ . The following facts make composite bundles useful for physical applications [38, 76, 83].

Given a composite bundle (51), let h be a global section of  $\Sigma \to X$ . Then the restriction

$$Y_h = h^* Y \tag{52}$$

of the fibre bundle  $Y \to \Sigma$  to  $h(X) \subset \Sigma$  is a subbundle of the fibre bundle  $Y \to X$ .

Every section s of the fibre bundle  $Y \to X$  is a composition of the section  $h = \pi_{Y\Sigma} \circ s$  of the fibre bundle  $\Sigma \to X$  and some section of the fibre bundle  $Y \to \Sigma$  over  $h(X) \subset \Sigma$ .

Let  $J^1\Sigma$ ,  $J^1_\Sigma Y$ , and  $J^1Y$  be jet manifolds of the fibre bundles  $\Sigma \to X$ ,  $Y \to \Sigma$  and  $Y \to X$ , respectively. They are provided with the adapted coordinates  $(x^\lambda, \sigma^m, \sigma^m_\lambda)$ ,  $(x^\lambda, \sigma^m, y^i, \tilde{y}^i_\lambda, y^i_m)$  and  $(x^\lambda, \sigma^m, y^i, \sigma^m_\lambda, y^i_\lambda)$ . There is the canonical map

$$\varrho: J^1\Sigma \underset{\Sigma}{\times} J^1\Sigma Y \xrightarrow{Y} J^1Y, \qquad y^i_{\lambda} \circ \varrho = y^i_m \sigma^m_{\lambda} + \widetilde{y}^i_{\lambda}.$$

Due to this map, any pair of connections

$$A_{\Sigma} = dx^{\lambda} \otimes (\partial_{\lambda} + A_{\lambda}^{i} \partial_{i}) + d\sigma^{m} \otimes (\partial_{m} + A_{m}^{i} \partial_{i}),$$

$$\Gamma = dx^{\lambda} \otimes (\partial_{\lambda} + \Gamma_{\lambda}^{m} \partial_{m})$$

$$(53)$$

on fibre bundles  $Y \to \Sigma$  and  $\Sigma \to X$ , respectively, yields the composite connection

$$\gamma = A_{\Sigma} \circ \Gamma = dx^{\lambda} \otimes (\partial_{\lambda} + \Gamma_{\lambda}^{m} \partial_{m} + (A_{\lambda}^{i} + A_{m}^{i} \Gamma_{\lambda}^{m}) \partial_{i})$$
 (54)

on the fibre bundle  $Y \to X$ . For instance, let us consider a vector field  $\tau$  on the base X, its horizontal lift  $\Gamma \tau$  onto  $\Sigma$  by means of the connection  $\Gamma$  and, in turn, the horizontal lift  $A_{\Sigma}(\Gamma \tau)$  of  $\Gamma \tau$  onto Y by means of the connection  $A_{\Sigma}$ . Then  $A_{\Sigma}(\Gamma \tau)$  is the horizontal lift of  $\tau$  onto Y by means of the composite connection  $\gamma$  (54).

Given a composite bundle Y (51), there is the exact sequence of bundles

$$0 \to V_{\Sigma}Y \to VY \to Y \underset{\Sigma}{\times} V\Sigma \to 0, \tag{55}$$

where  $V_{\Sigma}Y$  is the vertical tangent bundle of the fibre bundle  $Y \to \Sigma$ . Every connection A (53) on the fibre bundle  $Y \to \Sigma$  yields the splitting

$$\dot{y}^i \partial_i + \dot{\sigma}^m \partial_m = (\dot{y}^i - A_m^i \dot{\sigma}^m) \partial_i + \dot{\sigma}^m (\partial_m + A_m^i \partial_i)$$

of the exact sequences (55). This splitting defines the first order differential operator

$$\widetilde{D} = dx^{\lambda} \otimes (y_{\lambda}^{i} - A_{\lambda}^{i} - A_{m}^{i} \sigma_{\lambda}^{m}) \partial_{i}$$
(56)

on the composite bundle  $Y \to X$ . This operator, called the vertical covariant differential, possesses the following important property. Let h be a section of the fibre bundle  $\Sigma \to X$  and  $Y_h$  the subbundle (52) of the composite bundle  $Y \to X$ . Then the restriction of the vertical covariant differential  $\widetilde{D}$  (56) to  $J^1Y_h \subset J^1Y$  coincides with the familiar covariant differential relative to the pull-back connection

$$A_h = h^* A_{\Sigma} = dx^{\lambda} \otimes [\partial_{\lambda} + ((A_m^i \circ h)\partial_{\lambda} h^m + (A \circ h)_{\lambda}^i)\partial_i]$$
 (57)

on  $Y_h \to X$  [38, 59, 83].

The peculiarity of field theory on a composite bundle (51) is that its Lagrangian depends on a connection on  $Y \to \Sigma$ , but not  $Y \to X$ , and it factorizes through the vertical covariant differential (56). This is the case of field theories with broken symmetries, spinor fields, gauge gravitation theory [59, 69, 71, 73, 74, 83].

## 13 Symmetry breaking and Higgs fields

In gauge theory on a principal bundle  $P \to X$ , a symmetry breaking is defined as reduction of the structure Lie group G of this principal bundle to a closed (consequently, Lie) subgroup H of exact symmetries [22, 38, 54, 63, 67, 74, 83].

By virtue of the well-known theorem, reduction of the structure group of a principal bundle takes place iff there exists a global section h of the quotient bundle  $P/H \to X$ . This section is treated as a Higgs field. Thus, we have the composite bundle

$$P \to P/H \to X,$$
 (58)

where  $P \to P/H$  is a principal bundle with the structure group H and  $\Sigma = P/H \to X$  is a P-associated fibre bundle with the typical fibre G/H. Moreover, there is one-to-one correspondence between the global sections h of  $\Sigma \to X$  and reduced H-principal subbundles  $P^h = \pi_{P\Sigma}^{-1}(h(X))$  of P.

Let  $Y \to \Sigma$  be a vector bundle associated to the H-principal bundle  $P \to \Sigma$ . Then sections of the composite bundle  $Y \to \Sigma \to X$  describe matter fields with the exact symmetry group H in the presence of Higgs fields. Given bundle coordinates  $(x^{\lambda}, \sigma^m, y^i)$  on Y, these sections are locally represented by pairs  $(\sigma^m(x), y^i(x))$ . Given a global section h of  $\Sigma \to X$ , sections of the vector bundle  $Y_h$  (52) describe matter fields in the presence of the background Higgs field h. Moreover, for different Higgs fields h and h', the fibre bundles  $Y_h$  and  $Y_{h'}$  need not be equivalent [38, 67, 74].

Note that  $Y \to X$  fails to be associated to a principal bundle  $P \to X$  with the structure group G and, consequently, it need not admit a principal connection. Therefore, one should consider a principal connection (53) on the fibre bundle  $Y \to \Sigma$ , and a Lagrangian on  $J^1Y$  factorizes through the vertical covariant differential  $\widetilde{D}$  (56). In the presence of a background Higgs field h, the restriction of  $\widetilde{D}$  to  $J^1Y_h$  coincides with the covariant differential relative to the pull-back connection (57) on  $Y_h \to X$ .

Riemannian and pseudo-Riemannian metrics on a manifold X exemplify classical Higgs fields. Let X be an oriented four-dimensional smooth manifold and LX the fibre bundle of linear frames in the tangent spaces to X. It is a principal bundle with the structure group  $GL_4 = GL^+(4,\mathbb{R})$ . This structure group is always reducible to its maximal compact subgroup SO(4). The corresponding global sections of the quotient bundle LX/SO(4) are Riemannian metrics on X. However, the reduction of the structure group  $GL_4$  of LX to its Lorentz subgroup SO(1,3) and a pseudo-Riemannian metric on X need not exist.

Note that, if  $G = GL_4$  and H = SO(1,3), we are in the case of so called reductive G-structure [45] when the Lie algebra  $\mathfrak{g}$  of G is the direct sum

$$\mathcal{G} = \mathfrak{h} \oplus \mathfrak{m} \tag{59}$$

of the Lie algebra  $\mathfrak{h}$  of H and a subspace  $\mathfrak{m} \subset \mathfrak{g}$  such that  $ad(g)(\mathfrak{m}) \subset \mathfrak{m}$ ,  $g \in H$ . In this case, the pull-back of the  $\mathfrak{h}$ -valued component of any principal connection on P onto a reduced subbundle  $P^h$  is a principal connection on  $P^h$ .

# 14 Natural and gauge-natural bundles

A connection  $\Gamma$  on a fibre bundle  $Y \to X$  defines the horizontal lift  $\Gamma \tau$  onto Y of any vector field  $\tau$  on X. There is the category of natural bundles [56, 80] which admit the functorial lift  $\tilde{\tau}$  onto T of any vector field  $\tau$  on X such that  $\tau \mapsto \overline{\tau}$  is a monomorphism of the Lie algebra of vector field on X to that on T. One can think of the lift  $\tilde{\tau}$  as being an infinitesimal generator of a local one-parameter group of general covariant transformations of T. The corresponding Noether current  $\mathfrak{J}_{\tilde{\tau}}$  is the energy-momentum flow along  $\tau$  [38, 68, 69, 83].

Natural bundles are exemplified by tensor bundles over X. Moreover, all bundles associated to the principal frame bundle LX are natural bundles. The bundle

$$C_K = J^1 L X / G L_4 \tag{60}$$

of principal connections on LX is not associated to LX, but it is also a natural bundle [38, 59].

Note that a spinor bundle  $S^g$  associated to a pseudo-Riemannian metric g on X admits the canonical lift of any vector field on X onto  $S^h$ . It is called Kosmann's Lie derivative

[28, 46]. Such a lift is a property of any reductive G-structure [45]. However, this lift fails to be functorial, and spinor bundles are not natural.

In a more general setting, higher order natural bundles and gauge-natural bundles are called into play [27, 29, 56, 80]. Note that the linear frame bundle LX over a manifold X is the set of first order jets of local diffeomorphisms of  $\mathbb{R}^n$  to X,  $n=\dim X$ , at the origin of  $\mathbb{R}^n$ . Accordingly, one considers r-order frame bundles  $L^rX$  of r-order jets of local diffeomorphisms of  $\mathbb{R}^n$  to X. Furthermore, given a principal bundle  $P \to X$  with a structure group G, the r-order jet bundle  $J^1P \to X$  of its sections fails to be a principal bundle. However, the product  $W^rP = L^rX \times J^rP$  is a principal bundle with the structure group  $W^r_nG$  which is a semi direct product of the group  $G^r_n$  of invertible r-order jets of maps  $\mathbb{R}^n$  to itself at its origin (e.g.,  $G^1_n = GL(n,\mathbb{R})$ ) and the group  $T^r_nG$  of r-order jets of morphisms  $\mathbb{R}^n \to G$  at the origin of  $\mathbb{R}^n$ . Moreover, if  $Y \to X$  is a fibre bundle associated to P, the jet bundle  $J^rY \to X$  is a vector bundle associated to the principal bundle  $W^rP$ . It exemplifies gauge natural bundles, which can described as fibre bundles associated to principal bundles  $W^rP$ . Natural bundles are gauge natural bundles for a trivial G = 1. The bundle of principal connections C (39) is a first order gauge natural bundle. This fact motivates somebody to develop generalized gauge theory on gauge natural bundles [17, 23, 29, 30].

## 15 Gauge gravitation theory

Gauge gravitation theory (see [3, 4, 24, 50, 53, 55, 64, 73, 82] for a survey) can be described as a field theory on natural bundles over an oriented four-dimensional manifold X whose dynamic variables are linear connections and pseudo-Riemannian metrics on X [10, 59, 71, 73, 83].

Linear connections on X (henceforth world connection) are principal connections on the linear frame bundle LX of X. They are represented by sections of the bundle of linear connections  $C_K$  (60). This is provided with bundle coordinates  $(x^{\lambda}, k_{\lambda}{}^{\nu}{}_{\alpha})$  such that components  $k_{\lambda}{}^{\nu}{}_{\alpha} \circ K = K_{\lambda}{}^{\nu}{}_{\alpha}$  of a section K of  $C_K \to X$  are coefficient of the linear connection

$$K = dx^{\lambda} \otimes (\partial_{\lambda} + K_{\lambda}{}^{\mu}{}_{\nu}\dot{x}^{\nu}\dot{\partial}_{\mu})$$

on TX with respect to the holonomic bundle coordinates  $(x^{\lambda}, \dot{x}^{\lambda})$ .

In order to describe gravity, let us assume that the linear frame bundle LX admits a Lorentz structure, i.e., reduced principal subbundles with the structure Lorentz group. Global sections of the corresponding quotient bundle

$$\Sigma_{\rm PR} = LX/SO(1,3) \to X \tag{61}$$

are pseudo-Riemannian (henceforth world) metrics on X. This fact motivates us to treat a metric gravitational field as a Higgs field [53, 71, 73].

The total configuration space of gauge gravitation theory in the absence of matter fields is the bundle product  $\Sigma_{PR} \times C_K$  coordinated by  $(\sigma^{\alpha\beta}, k_{\mu}{}^{\alpha}{}_{\beta})$ . This is a natural bundle admitting the functorial lift

$$\widetilde{\tau}_{K\Sigma} = \tau^{\mu} \partial_{\mu} + (\sigma^{\nu\beta} \partial_{\nu} \tau^{\alpha} + \sigma^{\alpha\nu} \partial_{\nu} \tau^{\beta}) \frac{\partial}{\partial \sigma^{\alpha\beta}} + (\partial_{\nu} \tau^{\alpha} k_{\mu}{}^{\nu}{}_{\beta} - \partial_{\beta} \tau^{\nu} k_{\mu}{}^{\alpha}{}_{\nu} - \partial_{\mu} \tau^{\nu} k_{\nu}{}^{\alpha}{}_{\beta} + \partial_{\mu\beta} \tau^{\alpha}) \frac{\partial}{\partial k_{\mu}{}^{\alpha}{}_{\beta}}$$
(62)

of vector fields  $\tau$  on X [10, 59, 83]. These lifts are generators of one-dimensional groups of general covariant transformations, whose gauge parameters are vector fields on X.

We do not specify a gravitation Lagrangian  $L_G$  on the jet manifold  $J^1(\Sigma_{PR} \times C_K)$ , but assume that vector fields (62) exhaust its gauge symmetries. Then the Euler–Lagrange operator  $(\mathcal{E}_{\alpha\beta}d\sigma^{\alpha\beta} + \mathcal{E}^{\mu}{}_{\alpha}{}^{\beta}dk_{\mu}{}^{\alpha}{}_{\beta}) \wedge \omega$  of this Lagrangian obeys irreducible Noether identities

$$-(\sigma_{\lambda}^{\alpha\beta} + 2\sigma_{\nu}^{\nu\beta}\delta_{\lambda}^{\alpha})\mathcal{E}_{\alpha\beta} - 2\sigma^{\nu\beta}d_{\nu}\mathcal{E}_{\lambda\beta} + (-k_{\lambda\mu}{}^{\alpha}{}_{\beta} - k_{\nu\mu}{}^{\nu}{}_{\beta}\delta_{\lambda}^{\alpha} + k_{\beta\mu}{}^{\alpha}{}_{\lambda} + k_{\mu\lambda}{}^{\alpha}{}_{\beta})\mathcal{E}^{\mu}{}_{\alpha}{}^{\beta} + (-k_{\mu}{}^{\nu}{}_{\beta}\delta_{\lambda}^{\alpha} + k_{\mu}{}^{\alpha}{}_{\lambda}\delta_{\beta}^{\nu} + k_{\lambda}{}^{\alpha}{}_{\beta}\delta_{\nu}^{\nu})d_{\nu}\mathcal{E}^{\mu}{}_{\alpha}{}^{\beta} + d_{\mu\beta}\mathcal{E}^{\mu}{}_{\lambda}{}^{\beta} = 0$$

[10]. Taking the vertical part of vector fields  $\tilde{\tau}_{K\Sigma}$  and replacing gauge parameters  $\tau^{\lambda}$  with ghosts  $c^{\lambda}$ , we obtain the gauge operator and its nilpotent BRST prolongation

$$u_{E} = u^{\alpha\beta} \frac{\partial}{\partial \sigma^{\alpha\beta}} + u_{\mu}{}^{\alpha}{}_{\beta} \frac{\partial}{\partial k_{\mu}{}^{\alpha}{}_{\beta}} + u^{\lambda} \frac{\partial}{\partial c^{\lambda}} = (\sigma^{\nu\beta} c_{\nu}^{\alpha} + \sigma^{\alpha\nu} c_{\nu}^{\beta} - c^{\lambda} \sigma_{\lambda}^{\alpha\beta}) \frac{\partial}{\partial \sigma^{\alpha\beta}} + (c_{\nu}^{\alpha} k_{\mu}{}^{\nu}{}_{\beta} - c_{\beta}^{\nu} k_{\mu}{}^{\alpha}{}_{\nu} - c_{\mu}^{\nu} k_{\nu}{}^{\alpha}{}_{\beta} + c_{\mu\beta}^{\alpha} - c^{\lambda} k_{\lambda\mu}{}^{\alpha}{}_{\beta}) \frac{\partial}{\partial k_{\nu}{}^{\alpha}{}_{\beta}} + c_{\mu}^{\lambda} c^{\mu} \frac{\partial}{\partial c^{\lambda}},$$

but this differs from that in [49]. Accordingly, an original Lagrangian  $L_G$  is extended to a solution of the master equation

$$L_E = L_G + u^{\alpha\beta} \overline{\sigma}_{\alpha\beta} \omega + u_{\mu}^{\ \alpha}{}_{\beta} \overline{k}^{\mu}{}_{\alpha}{}^{\beta} \omega + u^{\lambda} \overline{c}_{\lambda} \omega,$$

where  $\overline{\sigma}_{\alpha\beta}$ ,  $\overline{k}^{\mu}{}_{\alpha}{}^{\beta}$  and  $\overline{c}_{\lambda}$  are corresponding antifields.

# 16 Dirac spinor fields

Dirac spinors as like as other ones are described in the Clifford algebra terms [33, 58]. The Dirac spinor structure on a four-dimensional manifold X is defined as a pair  $(P^h, z_s)$  of a principal bundle  $P^h \to X$  with the structure spin group  $L_s = SL(2, \mathbb{C})$  and its bundle morphism  $z_s : P^h \to LX$  to the frame bundle LX [6, 58]. Any such morphism factorizes

$$P^h \to L^h X \to L X \tag{63}$$

through some reduced principal subbundle  $L^hX \subset LX$  with the structure proper Lorentz group  $L = SO^{\uparrow}(1,3)$ , whose universal two-fold covering is  $L_s$ . The corresponding quotient bundle  $\Sigma_T = LX/L$  is a two-fold covering of the bundle  $\Sigma_{PR}$  (61). Its global section, called a tetrad field, defines a principal Lorentz subbundle  $L^hX$  of LX. It can be represented by a family of local sections  $\{h_a\}_{\iota}$  of LX on trivialization domains  $U_{\iota}$  which take values in  $L^hX$ and possess Lorentz transition functions. They define an atlas  $\Psi^h = \{(\{h_a\}_{\iota}, U_{\iota})\}$  of LXwith Lorentz transition functions such that the corresponding pseudo-Riemannian metric on X reads  $g_{\mu\nu} = h^a_{\nu}h^b_{\nu}\eta_{ab}$ , where  $\eta_{ab}$  is the Minkowski metric.

Thus, any Dirac spinor structure is associated to a Lorentz reduced structure, but the converse need not be true. There is the well-known topological obstruction to the existence of a Dirac spinor structure. For instance, a Dirac spinor structure on a non-compact manifold X exists iff X is parallelizable.

Given a Dirac spinor structure (63), the associated Dirac spinor bundle  $S^h$  can be seen as a subbundle of the bundle of Clifford algebras generated by the Lorentz frames  $\{t_a\} \in L^h X$  [16, 58]. This fact enables one to define the Clifford representation

$$\gamma_h(dx^\mu) = h_a^\mu \gamma^a \tag{64}$$

of coframes  $dx^{\mu}$  in the cotangent bundle  $T^*X$  by Dirac's matrices, and introduce the Dirac operator on  $S^h$  with respect to a principal connection on  $P^h$ . Then sections of a spinor bundle  $S^h$  describe Dirac spinor fields in the presence of a tetrad field h. However, the representations (64) for different tetrad fields fail to be equivalent. Therefore, one meets a problem of describing Dirac spinor fields in the presence of different tetrad fields and under general covariant transformations.

In order to solve this problem, let us consider the universal two-fold covering  $\widetilde{GL}_4$  of the group  $GL_4$  and the  $\widetilde{GL}_4$ -principal bundle  $\widetilde{LX} \to X$  which is the two-fold covering bundle of the frame bundle LX [26, 58, 77]. Then we have the commutative diagram

$$\widetilde{LX} \xrightarrow{\zeta} LX$$

$$\uparrow \qquad \uparrow$$

$$P^h \longrightarrow L^h X$$

for any Dirac spinor structure (63) [34, 69, 71]. As a consequence,  $\widetilde{LX}/\mathrm{L}_s = LX/L = \Sigma_{\mathrm{T}}$ . Since  $\widetilde{LX} \to \Sigma_T$  is an  $\mathrm{L}_s$ -principal bundle, one can consider the associated spinor bundle  $S \to \Sigma_T$  whose typical fibre is a Dirac spinor space  $V_s$  [59, 69, 71, 83]. We agree to call it the universal spinor bundle because, given a tetrad field h, the pull-back  $S^h = h^*S \to X$  of S onto X is a spinor bundle on X which is associated to the  $\mathrm{L}_s$ -principal bundle  $P^h$ . The universal spinor bundle S is endowed with bundle coordinates  $(x^\lambda, \sigma_a^\mu, y^A)$ , where  $(x^\lambda, \sigma_a^\mu)$  are bundle coordinates on  $\Sigma_T$  and  $y^A$  are coordinates on the spinor space  $V_s$ . The universal

spinor bundle  $S \to \Sigma_T$  is a subbundle of the bundle of Clifford algebras which is generated by the bundle of Minkowski spaces associated to the L-principal bundle  $LX \to \Sigma_T$ . As a consequence, there is the Clifford representation

$$\gamma_{\Sigma}: T^*X \underset{\Sigma_T}{\otimes} S \to S, \qquad \gamma_{\Sigma}(dx^{\lambda}) = \sigma_a^{\lambda} \gamma^a,$$
 (65)

whose restriction to the subbundle  $S^h \subset S$  restarts the representation (64).

Sections of the composite bundle  $S \to \Sigma_T \to X$  describe Dirac spinor fields in the presence of different tetrad fields as follows [69, 71]. Due to the splitting (59), any general linear connection K on X (i.e., a principal connection on LX) yields the connection

$$A_{\Sigma} = dx^{\lambda} \otimes (\partial_{\lambda} - \frac{1}{4} (\eta^{kb} \sigma_{\mu}^{a} - \eta^{ka} \sigma_{\mu}^{b}) \sigma_{k}^{\nu} K_{\lambda}{}^{\mu}{}_{\nu} L_{ab}{}^{A}{}_{B} y^{B} \partial_{A}) + d\sigma_{k}^{\mu} \otimes (\partial_{\mu}^{k} + \frac{1}{4} (\eta^{kb} \sigma_{\mu}^{a} - \eta^{ka} \sigma_{\mu}^{b}) L_{ab}{}^{A}{}_{B} y^{B} \partial_{A})$$

$$(66)$$

on the universal spinor bundle  $S \to \Sigma_T$ . Its restriction to  $S^h$  is the familiar spin connection

$$K_h = dx^{\lambda} \otimes [\partial_{\lambda} + \frac{1}{4} (\eta^{kb} h^a_{\mu} - \eta^{ka} h^b_{\mu}) (\partial_{\lambda} h^{\mu}_{k} - h^{\nu}_{k} K_{\lambda}{}^{\mu}{}_{\nu}) L_{ab}{}^{A}{}_{B} y^{B} \partial_{A}], \tag{67}$$

defined by K [66, 68]. The connection (66) yields the vertical covariant differential

$$\widetilde{D} = dx^{\lambda} \otimes [y_{\lambda}^{A} - \frac{1}{4} (\eta^{kb} \sigma_{\mu}^{a} - \eta^{ka} \sigma_{\mu}^{b}) (\sigma_{\lambda k}^{\mu} - \sigma_{k}^{\nu} K_{\lambda}^{\mu}{}_{\nu}) L_{ab}{}^{A}{}_{B} y^{B}] \partial_{A}, \tag{68}$$

on the fibre bundle  $S \to X$ . Its restriction to  $J^1S^h \subset J^1S$  recovers the familiar covariant differential on the spinor bundle  $S^h \to X$  relative to the spin connection (67). Combining (65) and (68) gives the first order differential operator

$$\mathcal{D} = \sigma_a^{\lambda} \gamma^{aB}{}_A [y_{\lambda}^A - \frac{1}{4} (\eta^{kb} \sigma_{\mu}^a - \eta^{ka} \sigma_{\mu}^b) (\sigma_{\lambda k}^{\mu} - \sigma_{k}^{\nu} K_{\lambda}{}^{\mu}{}_{\nu}) L_{ab}{}^A{}_B y^B],$$

on the fibre bundle  $S \to X$ . Its restriction to  $J^1S^h \subset J^1S$  is the familiar Dirac operator on the spinor bundle  $S^h$  in the presence of a background tetrad field h and a general linear connection K.

#### References

- [1] S.Al-Ashhab and R.Fulp, Canonical transformations of local functionals and Sh-Li structures, *J. Geom. Phys.* **53** (2005) 365.
- [2] V.Aldaya and J. de Azcárraga, Geometric formulation of classical mechanics and field theory, *Riv. Nuovo Cim.* **3** (1980) 1.

- [3] V.Aldaya and E.Sánches-Sastre, Gauge theories of gravity and mass generation, *Int. J. Geom. Methods Mod. Phys.* **5** (2008) 197.
- [4] S.Ali and S.Capozziello, Nonlinera realization mof the local conform-affine symmetry group for gravity in the composite fiber bundle formalism, *Int. J. Geom. Methods Mod. Phys.* 4 (2007) 1041.
- [5] I.Anderson, Introduction to the variational bicomplex, *Contemp. Math.* **132** (1992) 51.
- [6] S.Avis and C.Isham, Generalized spin structure on four dimensional space-times, Comm. Math. Phys. 72 (1980) 103.
- [7] G.Barnish, R.Fulp, T.Lada and J.Stasheff, The Sh-Lie structure of Poisson brackets in field theory, *Commun. Math. Phys.* **191** (1998) 585.
- [8] G.Barnich, F.Brandt and M.Henneaux, Local BRST cohomology in gauge theories, *Phys. Rep.* **338** (2000) 439.
- [9] C.Bartocci, U.Bruzzo and D.Hernández Ruipérez, *The Geometry of Supermanifolds* (Dordrecht: Kluwer Academic Publ., Dordrecht, 1991).
- [10] D.Bashkirov and G.Sardanashvily, On the BV quantization of gauge gravitation theory, Int. J. Geom. Methods Mod. Phys. 2 (2005) 203.
- [11] D.Bashkirov, G.Giachetta, L.Mangiarotti and G.Sardanashvily, Noether's second theorem in a general setting. Reducible gauge theories, *J. Phys. A* **38** (2005) 5329.
- [12] D.Bashkirov, G.Giachetta, L.Mangiarotti and G.Sardanashvily, Noether's second theorem for BRST symmetries, *J. Math. Phys.* **46** (2005) 053517.
- [13] D.Bashkirov, G.Giachetta, L.Mangiarotti and G.Sardanashvily, The antifield Koszul–Tate complex of reducible Noether identities, *J. Math. Phys.* **46** (2005) 103513.
- [14] D.Bashkirov, G.Giachetta, L.Mangiarotti and G.Sardanashvily, The KT-BRST complex of degenerate Lagrangian systems, *Lett. Math. Phys.* 83 (2008) 237-252.
- [15] M.Bauderon, Differential geometry and Lagrangian formalism in the calculus of variations, in *Differential Geometry, Calculus of Variations, and their Applications*, Lecture Notes in Pure and Applied Mathematics, 100 (Marcel Dekker, Inc., N.Y., 1985), p. 67.
- [16] I.Benn and T.Tucker, An Introduction to Spinors and Geometry with Applications to Physics (Adam Hilger, Bristol, 1987)

- [17] E.Bibbona, L.Fatibene, M.Francaviglia, Gauge-natural parameterized variational problems, vakonomic field theories and relativistic hydrodynamics of a charged fluid, *Int. J. Geom. Methods mod. Phys.* **3** (2006) 1573.
- [18] A.Borowoiec, L.Fatibene, M.Ferraris and M.Francaviglia, Covariant Lagrangian formulation of Chern-Simons and BF theories, Int. J. Geom. Methods Mod. Phys. 3 (2006) 755.
- [19] A.Borowoiec, L.Fatibene, M.Ferraris and S.Mercadante, Covariant Lagrangian formulation of Chern–Simons theories, *Int. J. Geom. Methods mod. Phys.* 4 (2007) 277.
- [20] R.Bryant, S.Chern, R.Gardner, H.Goldschmidt, P.Griffiths, Exterior Differential Systems (Springer-Verlag, Berlin, 1991).
- [21] J.Cariñena and H.Figueroa, Singular Lagrangian in supermechanics, *Diff. J. Geom.* Appl. 18 (2003) 33.
- [22] M.Castrillón-López and T.Ratiu, Reduction in principal bundles: covariant Lagrange–Poincaré equations, *Commun. Math. Phys.* **236** (2003) 223.
- [23] C.Castro, A Chern-Simons  $E_8$  gauge theory of gravity in D=15, grand unification and generalized gravity in Clifford spaces, Int. J. Geom. Methods Mod. Phys. 4 (2007) 1239.
- [24] A.Chamseddine, Application of the gauge principle to gravitational interactions, *Int. J. Geom. Methods Mod. Phys.* **3** (2006) 149.
- [25] R.Cianci, M.Francaviglia and I.Volovich, Variational calculus and Poincaré–Cartan formalism in supermanifolds, *J. Phys. A.* **28** (1995) 723.
- [26] L.Dabrowski and R.Percacci, Spinors and diffeomorphisms, Comm. Math. Phys. 106 (1986) 691.
- [27] D.Eck, Gauge-natural bundles and generalized gauge theories, *Mem. Amer. Math. Soc.* **247** (1981) 1.
- [28] L.Fatibene, M.Ferraris, M.Francaviglia and M.Godina, Gauge formalism for General Relativity and fermionic matter, *Gen. Rel. Grav.* **30** (1998) 1371.
- [29] L.Fatibene and M.Francaviglia, Natural and Gauge Natural Formalism for Classical Field Theories. A Geometric Perspective Including Spinors and Gauge Theories (Kluwer, Dordrecht, 2003).

- [30] L.Fatibene, M.Ferraris and M.Francaviglia, On the gauge natural structure of modern physics, *Int. J. Geom. Methods Mod. Phys.* **1** (2004) 443.
- [31] L.Fatibene, M.Francaviglia and S.Mercadante, Covariant formulation of Chern–Simons theories, *Int. J. Geom. Methods Mod. Phys.* **2** (2005) 993.
- [32] D.Franco and C.Polito, Supersymmetric field-theoretic models on a supermanifold, *J. Math. Phys.* **45** (2004) 1447.
- [33] T.Friedrich, *Dirac Operators in Riemannian Geometry* Graduate Studies in Mathematics **25** (Amer. Math. Soc., Providence, RI, 2000).
- [34] R.Fulp, J.Lawson and L.Norris, Geometric prequantization on the spin bundle based on *n*-symplectic geometry: the Dirac equation, *Int. J. Theor. Phys.* **33** (1994) 1011.
- [35] R.Fulp, T.Lada and J.Stasheff, Sh-Lie algebras induced by gauge transformations, *Commun. Math. Phys.* **231** (2002) 25.
- [36] R.Fulp, T.Lada, and J.Stasheff, Noether variational Theorem II and the BV formalism, Rend. Circ. Mat. Palermo (2) Suppl. No. 71 (2003) 115.
- [37] A.Fuster, M.Henneaux and A.Maas, BRST quantization: a short review, *Int. J. Geom. Methods Mod. Phys.* **2** (2005) 939.
- [38] G.Giachetta, L.Mangiarotti and G.Sardanashvily, New Lagrangian and Hamiltonian Methods in Field Theory (World Scientific, Singapore, 1997).
- [39] G.Giachetta, L.Mangiarotti and G.Sardanashvily, Iterated BRST cohomology, *Lett. Math. Phys.* **53** (2000) 143.
- [40] G.Giachetta, L.Mangiarotti and G.Sardanashvily, Cohomology of the infinite-order jet space and the inverse problem, *J. Math. Phys.* **42** (2001) 4272.
- [41] G.Giachetta, L.Mangiarotti and G.Sardanashvily, Noether conservation laws in higher-dimensional Chern–Simons theory, *Mod. Phys. Lett. A* **18** (2003) 2645.
- [42] G.Giachetta, L.Mangiarotti and G.Sardanashvily, Lagrangian supersymmetries depending on derivatives. Global analysis and cohomology. *Commun. Math. Phys.* **259** (2005) 103.
- [43] G.Giachetta, L.Mangiarotti and G.Sardanashvily, Geometric and Algebraic Topological Methods in Quantum Mechanics (World Scientific, Singapore, 2005).

- [44] G.Giachetta, L.Mangiarotti and G.Sardanashvily, On the notion of gauge symmetries of generic Lagrangian field theory, *J. Math. Phys* **50** (2009) 012903.
- [45] M.Godina and P.Matteucci, Reductive G-structure and Lie derivatives, J. Geom. Phys. 47 (2003) 66.
- [46] M.Godina and P.Matteucci, The Lie derivative of spinor fields: Theory and applications, Int. J. Geom. Methods Mod. Phys. 2 (2005) 159.
- [47] J.Gomis, J.París and S.Samuel, Antibracket, antifields and gauge theory quantization, *Phys. Rep* **295** (1995) 1.
- [48] M.Gotay, A multisymplectic framework for classical field theory and the calculus of variations. I. Covariant Hamiltonian formalism, in *Mechanics, Analysis and Geometry:* 200 Years after Lagrange, ed. M.Francaviglia (North Holland, Amsterdam, 1991), p. 203.
- [49] F.Gronwald, BRST antifield treatment of metric-affine gravity, *Phys. Rev. D* **57** (1998) 961.
- [50] F.Hehl, J.McCrea, E.Mielke and Y.Ne'eman, Metric-affine gauge theory of gravity: field equations, Noether identities, world spinors, and breaking of dilaton invariance, *Phys. Rep.* **258** (1995) 1.
- [51] R.Hermann, Differential Geometry and the Calculus of Variations (Academic Press, Brookline, 1968).
- [52] F.Hirzebruch, Topological Methods in Algebraic Geometry (Springer, Berlin, 1966).
- [53] D.Ivanenko and G.Sardanashvily, The gauge treatment of gravity, *Phys. Rep.* **94** (1983) 1.
- [54] M.Keyl, About the geometric structure of symmetry breaking, *J. Math. Phys.* **32** (1991) 1065.
- [55] J.Klauder, Overview of affine quantum gravity, Int. J. Geom. Methods Mod. Phys. 3 (2006) 81.
- [56] I.Kolář, P.Michor and J.Slovák, *Natural Operations in Differential Geometry* (Springer-Verlag, Berlin, 1993).
- [57] I.Krasil'shchik, V.Lychagin and A.Vinogradov, Geometry of Jet Spaces and Nonlinear Partial Differential Equations (Gordon and Breach, Glasgow, 1985).

- [58] H.Lawson, Jr., and M.-L.Michelsohn, *Spin Geometry* (Princeton Univ. Press., Princeton, 1989).
- [59] L.Mangiarotti and G.Sardanashvily, Connections in Classical and Quantum Field Theory (World Scientific, Singapore, 2000).
- [60] J.Martín and A.Tiemblo, The role of translational invariance in nonlinear gauge theories of gravity, *Int. J. Geom. Methods Mod. Phys.* **5** (2008) 253.
- [61] J.Monterde, J.Masqué and J.Vallejo, The Poincaré–Cartan form in superfield theory, *Int. J. Geom. Methods Mod. Phys.* **3** (2006) 775.
- [62] M.Nieper-Wisskirchen, Operads and jet modules, Int. J. Geom. Methods Mod. Phys 2 (2005) 1133.
- [63] L.Nikolova and V.Rizov, Geometrical approach to the reduction of gauge theories with spontaneous broken symmetries, *Rep. Math. Phys.* **20** (1984) 287.
- [64] Yu.Obukhov, Poincaré gauge gravity: Selected topics, Int. J. Geom. Methods Mod. Phys. 3 (2006) 95.
- [65] P.Olver, Applications of Lie Groups to Differential Equations (Springer-Verlag, Berlin, 1986).
- [66] V.Ponomarev and Yu.Obukhov, Generalized Einstein-Maxwell theory, Gen. Rel. Grav. 14 (1982) 309.
- [67] G.Sardanashvily, On the geometry of spontaneous symmetry breaking, *J. Math. Phys.* **33** (1992) 1546.
- [68] G.Sardanashvily, Stress-energy-momentum conservation law in gauge gravitation theory, Class. Quant. Grav. 14 (1997) 1371.
- [69] G.Sardanashvily, Covariant spin structure, J. Math. Phys. 39 (1998) 4874.
- [70] G.Sardanashvily, Cohomology of the variational complex in the class of exterior forms of finite jet order, *Int. J. Math. and Math. Sci.* **30** (2002) 39.
- [71] G.Sardanashvily, Classical gauge theory of gravity, *Theor. Math. Phys.* **132** (2002) 1163.
- [72] G.Sardanashvily, Noether identities of a differential operator. The Koszul-Tate complex. Int. J. Geom. Methods Mod. Phys. 2 (2005) 873.

- [73] G.Sardanashvily, Gauge gravitation theory from geometric viewpoint, *Int. J. Geom. Methods Mod. Phys.* **3** (2006) N1, v-xx.
- [74] G.Sardanashvily, Geometry of classical Higgs fields, *Int. J. Geom. Methods Mod. Phys.* **3** (2006) 139.
- [75] G.Sardanashvily, Graded infinite order jet manifolds, Int. J. Geom. Methods Mod. Phys. 4 (2007) 1335.
- [76] D.Saunders, *The Geometry of Jet Bundles*, London Math. Soc., Lecture Note Series **142** (Cambr. Univ. Press, Cambridge, 1989).
- [77] S.Switt, Natural bundles. II. Spin and the diffeomorphism group, *J. Math. Phys.* **34** (1993) 3825.
- [78] F.Takens, Symmetries, conservation laws and variational principles, in *Geometry and Topology*, Lect. Notes Math., **597** (Springer-Verlag, Berlin, 1977), p. 581.
- [79] F.Takens, A global version of the inverse problem of the calculus of variations, *J. Diff. Geom.* **14** (1979) 543.
- [80] C.Terng, Natural vector bundles and natural differential operators, American J. Math. 100 (1978) 775.
- [81] W.Tulczyiew, The Euler-Lagrange resolution, in *Differential Geometric Methods in Mathematical Physics (Proc. Conf., Aix-en-Provence/Salamanca, 1979)* Lect. Notes Math. **836** (Springer-Verlag, Berlin, 1980), p. 22.
- [82] S.Vignolo, R. Cianci and D.Bruno, General relativity as a constrained gauge theory, *Int. J. Geom. Methods Mod. Phys.* **3** (2006) 1493.
- [83] G.Giachetta, L.Mangiarotti and G.Sardanashvily, Advanced Classical Field Theory (World Scientific, Singapore, 2009).